

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1262

TURBULENCE AND HEAT STRATIFICATION

By H. Schlichting

Translation of "Turbulenz bei Wärmeschichtung."
Zeitschrift für Angewandte Mathematik und Mechanik,
Band 15, Heft 6, December 1935



Washington
October 1950



NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1262

TURBULENCE AND HEAT STRATIFICATION*

By H. Schlichting

SUMMARY

In this report the method of small oscillations is used to investigate the stability of a stratified plane laminar flow (x and y are rectangular coordinates in the plane; x direction = primary flow direction, y direction perpendicular upward). The laminar flow $U = U(y)$ serving as basis is a "boundary-layer flow" which increases from the value $U = 0$ at the wall to a constant value $U = U_m$ at great wall distances. Stratification of densities is assumed in the boundary layer (thickness = δ), while outside of the boundary layer, in the zone of constant velocity U_m , the density is assumed to be constant. The flow function of a partial oscillation of the superposed periodic disturbance motion is written as

$$\psi(x, y, t) = \varphi(y)e^{i(\alpha x - \beta t)} = \varphi(y)e^{i\alpha(x - ct)}$$

where α is always real and identifies the spatial angular frequency of the disturbance ($\lambda = 2\pi/\alpha$ = wave length of the disturbance); β is, in general, complex, and the prefix of the imaginary part of β decides the stability or instability of the disturbance. The stability investigation is a characteristic value problem of the differential equation of the disturbance

$$\begin{aligned} (U - c)^2 [\varphi'' - \alpha^2 \varphi] - (U - c) U'' \varphi + K \varphi + L(U - c) [- (U - c) \varphi' + U' \varphi] \\ = -\frac{1}{\alpha R} (U - c) [\varphi''' - 2\alpha^2 \varphi'' + \alpha^4 \varphi - L(\varphi'' - \alpha^2 \varphi')] \end{aligned}$$

for the disturbance amplitude φ , with the boundary conditions $\varphi = \varphi' = 0$ at the wall $y = 0$, and $y = \infty$ at great distance from the wall;

*"Turbulenz bei Wärmeschichtung." Zeitschrift für Angewandte Mathematik und Mechanik, Band 15, Heft 6, December 1935, pp. 313-338. Lecture presented at the fourth international Mechanics Conference held at Cambridge, England, from July 3 to July 9, 1934.

R denotes the Reynolds number, $K = - \frac{\delta}{\rho} \frac{dp}{dy} \frac{g\delta}{U_m^2}$ (Richardson number)

and $L = - \frac{\delta}{\rho} \frac{dp}{dy}$ are nondimensional stratification numbers, $\sqrt{\frac{L}{K}} = U_m / \sqrt{\delta g}$

is a Froude number. For a specified group of four values α , R , K , L , this equation has only one solution for a particular value of c . The study is limited to pure real values of c , that is, to the determination of the curve of the transition points from stability to instability (indifference curve) for specified values of K and L in the αR plane. Stable ($K > 0$, $L > 0$) and unstable stratifications ($K < 0$, $L < 0$) are investigated. It is found that for constant Froude number with increasing stratification (increasing K) the critical Reynolds number is greater and the region of unstable disturbances in the αR plane smaller until finally at a critical value of the stratification quantity K , which still depends on the Froude number, complete stability of flow prevails for all disturbance wave lengths and Reynolds numbers.

The calculation is carried out for the flow past a flat plate, for which Tollmien had computed the indifference curve for homogeneous flow.

The critical stratification quantity $\Theta_K = \frac{g}{\rho} \frac{dp}{dy} \left/ \left(\frac{dU}{dy} \right)_w^2 \right.$, at which the

turbulence must become zero, ranges from 0.0409 to 0.029 for Froude

numbers $\frac{U_m^2}{\delta g} = 0$ to 5, while Taylor and Goldstein arrived at $\Theta_K = 1/4$ by

a similar calculation in which the fluid friction and the curvature of the profile had been neglected, and Richardson and Prandtl had obtained $\Theta_K = 1$ and $\Theta_K = 2$, respectively, by rough estimates.

The essential premises of the present investigation (plate profile, stratification of densities in boundary layer only) are fairly well confirmed by Reichardt's measurements in the Göttingen hot-cold air tunnel, where the upper plate is heated with steam, and the lower plate cooled with tap water (stable case). The decision, whether a measured velocity profile was laminar or turbulent, was made from oscillographic records of the voltage fluctuations of a hot wire. The comparison of the measurements with the present theory indicates a very satisfactory agreement.

1. INTRODUCTION

On cool summer evenings, when there is a slight breeze, a person can occasionally observe the floating of damp fog with a very sharply

defined boundary over a wet meadow, a sign of the fact that the turbulence of the wind has stopped completely, so that the air layers slide laminar over one another without turbulent intermingling. This is due to the development of a marked temperature difference as a result of the evening cooling, which prevents the warmer and hence specifically lighter upper layer from mixing with the cooler, heavier air layers near the ground.

Closely related with it are Richardson's observations (reference 1) of the influence of the gustiness of the wind which can be regarded as a measure for the turbulence strength on the vertical temperature difference. He observed that the gustiness is small when the air is colder below than above and that it increases when the temperature difference approaches the unstable adiabatic equilibrium. But, surprisingly, the gustiness does not increase much more when the unstable superadiabatic temperature difference prevails.

Taylor had made similar observations in 1916 (reference 2). From simultaneous records of temperature and wind velocity at about 40 meters height above ground for several days and nights, he found that on the nights when the nocturnal temperature minimum was very marked as a result of strong radiation, the turbulence was completely gone. But on nights with lesser cooling, such as result from cloudiness, the wind velocity fluctuations were almost as great as during daytime.

The flow of fresh water over salt water without substantial intermingling, as observed in the Kattegat, for instance, belongs to the same group of phenomena. Even the surprising stability of Bjerknes's polar fronts, where the cold air masses form a wedge under the warm masses, is traceable to the stratification (reference 3). Taylor, in 1927, made a simple experiment by which the stabilizing effect of the stratification of densities on the turbulence can be shown by means of a salt solution (reference 4).

Conversely, there is increased turbulence, and hence a stronger intermingling, as a result of convection motions; and, when following strong radiation, the lower air layers are heated more than the upper layers (reference 5).

Reichardt has conducted experimental investigations on a stratified flow since 1927, under the direction of Professor Prandtl. A stream of air in a horizontal channel is blown between a plate cooled with water and heated with steam, whereby the upper, as well as the lower, plate can be heated (stable or unstable stratification). In 1927, when these experiments were still in the initial stage, Prandtl set up a simple theorem, in form of an energy consideration (reference 6), which is briefly reviewed.

Assume a flow in horizontal direction, with a stratification of densities in perpendicular direction (y-direction), so that the density decreases continuously upward. During the turbulent mixing motions, work is then performed by the fact that heavier matter is raised and lighter matter lowered with respect to the lift. The path traversed by a particle in vertical direction, before it mixes again with the new surrounding, is the Prandtl mixing path l (reference 7), and the difference in lift per unit volume of a particle shifted in the vertical over the path length l is $-gl \frac{d\rho}{dy}$. The work per unit volume of displaced fluid mass while traveling over the path length l is accordingly

$$\int_{y=y_1}^{y=y_1+l} \left(-g(y - y_1) \frac{d\rho}{dy} \right) dy = -\frac{g}{2} l^2 \frac{d\rho}{dy}$$

To identify the quantities participating on the exchange, visualize a level surface F ; on one fraction β_1 of this area, an upward motion with velocity v_1 prevails, and on a fraction β_2 a downward motion with velocity v_2 , so that the total flow volume in unit time is $F(\beta_1 v_1 + \beta_2 v_2)$. With it, the lifting work due to the stratification of densities is

$$L_S = -F(\beta_1 v_1 + \beta_2 v_2) \frac{1}{2} g l^2 \frac{d\rho}{dy} \quad (1)$$

This work must be supplied from the stored energy of the turbulent mixing motions. This is obtained by the work of the basic flow at the element, which is given by the product of the apparent turbulent shearing stress with the displacement velocity. For a body of base area F and height l , in which the afore-mentioned volume is exchanged, this work is $F \tau l \frac{du}{dy}$; τ is the turbulent shearing stress for which $\tau = \overline{\rho u' v'}$, according to Reynolds, where u' and v' are the turbulent fluctuation velocities ($u = u(y)$ = mean flow velocity). According to Prandtl, $u' = l \frac{du}{dy}$ and, according to the foregoing, $v' = \beta_1 v_1 + \beta_2 v_2$. Up to a numerical factor κ , which is still to be determined, in which the correlation factor of $u' v'$ enters, the turbulent shearing stress is then

$$\tau = \kappa \rho l \frac{du}{dy} (\beta_1 v_1 + \beta_2 v_2)$$

hence the work L_t of the turbulent apparent friction

$$L_t = \kappa F \rho l^2 \left(\frac{du}{dy} \right)^2 (\beta_1 v_1 + \beta_2 v_2) \quad (2)$$

The existence and disappearance of the turbulence as a result of the stable stratification depends upon which of the two energy amounts is the greater. If the work of the turbulent apparent friction is greater ($L_t > L_s$), the difference for maintaining the turbulence remains, but, if the work against the weight difference is greater ($L_s > L_t$), the turbulence must die out. Turbulence is therefore possible on an energy basis when

$$F \kappa \rho l^2 \left(\frac{du}{dy} \right)^2 (\beta_1 v_1 + \beta_2 v_2) > - F (\beta_1 v_1 + \beta_2 v_2) \frac{g}{2} l^2 \frac{d\rho}{dy}$$

or, after abbreviation

$$\kappa \rho \left(\frac{du}{dy} \right)^2 > - \frac{g}{2} \frac{d\rho}{dy}$$

For the nondimensional quantity

$$\frac{- \frac{g}{\rho} \frac{d\rho}{dy}}{\left(\frac{du}{dy} \right)^2} = \Theta \quad (3)$$

which serves as basis for all flows with stratification of densities, the "Richardson number" is introduced, since he was the first to study stratified flows in 1920 (reference 9). Thus $\Theta = 0$ denotes the homogeneous fluid. By Prandtl's energy appraisal the result is as follows:

Turbulence, on an energy basis, is possible for $\Theta < 2\kappa$
 Turbulence, on an energy basis, is impossible for $\Theta > 2\kappa$

Although nothing definite can be said about the numerical factor κ , it is suspected that it lies near unity. Prandtl chose $\kappa = 1$ and so obtained $\Theta = 2$ as stability limit.

Taylor indicated subsequently (reference 10) that a factor 2 cancels out when Prandtl's considerations are refined, so that Prandtl's stability limit $\Theta = 1$ would agree with similar considerations by Richardson (reference 9). Taylor (reference 11) and Goldstein (reference 12) later continued the theoretical study of flow with stratification of densities, that is, the stability of a plane laminar flow with stable stratification of densities, by the method of small vibrations. Viscosity and compressibility were disregarded and the velocity profiles were limited to those consisting of straight pieces, for reasons of mathematical simplicity. In the differential equation of the disturbance, only the effects of the stratification of densities on the potential energy (gravity effect) were taken into account, while the inertia effect of the stratification of densities is neglected. The cases in which this is permitted, and the resulting simplification in the calculation, will be discussed later. The results of Taylor's and Goldstein's stability investigations can be represented in the two nondimensionals Θ and λ/δ , λ = wave length of disturbance, δ a characteristic length of the velocity profile (boundary-layer thickness). For the majority of cases explored by Taylor and Goldstein, no actual stability limit resulted for Θ , but there still remained a certain although very narrow range of unstable disturbance wave lengths λ for every value of Θ .

Taylor obtained a definite result for the case of a fluid extended infinitely upward or downward, with laminar velocity distribution and uniform density distribution; he obtained $\Theta = 1/4$ as stability limit. Goldstein arrived at the same stability limit for the case of a fluid extending to infinity upward or downward; with uniform density distribution, the velocity below and above is constant, but varies linearly with the height in an intermediate layer.

A comparison of this theoretical result with test data from the Göttingen warm-cold air tunnel produced no agreement at all relative to the stability limit $\Theta = 1/4$, where, of course, it should be borne in mind that the velocity profiles measured in the tunnel are not linear, as postulated in the Taylor-Goldstein theory. It therefore seems appropriate to study the modification of the Taylor-Goldstein calculations with due allowance for the friction and to choose for the velocity distribution of the laminar flow a profile better adapted to the conditions to be realized by experiment. This appeared to be all the more promising, as the stability study on the homogeneous flow itself produced a satisfactory result only when the friction was included in suitable manner.

Tollmien (reference 13) indicated that profiles with finite curvature other than zero must be used as basis, since the approximation of a laminar profile by pieces of straight lines is insufficient.

In the first stability investigations of the laminar flow of a homogeneous fluid by Lord Rayleigh (reference 14), both the friction and the profile curvature were disregarded; they did not give the looked-for instability. Further investigations by Sommerfeld (reference 15), R. v. Mises (reference 16), and L. Hopf (reference 17) on the Couette flow (linear velocity distribution), which partly included the friction, also failed to give the desired results; it resulted in stability for all disturbance wave lengths and all Reynolds numbers. Subsequent investigations by Prandtl (reference 18) and Tietjens (reference 19), who allowed for the maximum friction terms and used profiles consisting of pieces of straight lines as basis, obtained an instability for the first time, but still no stability limit. Complete success, that is, the correct theoretical calculation of the stability limit (critical Reynolds number) was bestowed on these investigations only after Tollmien took the curvature of the velocity profile also into consideration.

The Taylor-Goldstein stability studies of the flow of stratified fluids form the analogy with Rayleigh's study on homogeneous fluid, as they ignore friction, as well as profile curvature. They are to be extended in the following by taking friction and profile curvature into account. (Compare the subsequent outline.)

Outline of Past Stability Investigations

	Homogeneous fluid		Inhomogeneous fluid with stratification of densities	
	Without friction	With friction	Without friction	With friction
Linear profile	Rayleigh	Sommerfeld, v. Mises, Hopf, Prandtl, Tietjens et al	G. I. Taylor Goldstein	
Curved profile		Tollmien		Schlichting

Closely allied with these flows with stratification of densities are the curved flows of homogeneous fluid, as, for example, the stable stratification due to the centrifugal forces in the two-dimensional flow between two concentric, rotating cylinders, of which the one on the inside is stationary, while the one on the outside rotates. In 1927 Prandtl obtained, by an energy study similar to that on flow with stratification of densities, a stability limit for a flow stratified by centrifugal forces, which is in close agreement with Wendt's measurements (reference 20).

The effect of such a stable stratification of centrifugal forces on the critical Reynolds number was studied in an earlier report (reference 21), and the results were also in satisfactory accord with experiment.

Returning to the flow with stratification of densities, the stability investigation can very generally be formulated as follows:

For each superposed disturbance (wave length λ), the magnitude of amplification (damping) is to be computed for each Reynolds number and for each layer (Richardson number), the laminar flow $U = U(y)$ being specified. If A_0 is the amplitude of disturbance at time $t = 0$, the amplitude A at time t is $A = A_0 e^{\beta_1 t}$, where β_1 is a measure of the amplification. This problem is dependent on eight essential factors, namely,

β_1	log de(in)crement of the amplification
λ	wave length of disturbance
δ	characteristic length of laminar flow
U_m	maximum velocity of laminar flow
ρ	density
$\Delta\rho$	density difference in laminar flow
ν	kinematic viscosity
g	gravitational acceleration

From these eight quantities, five independent nondimensionals can be formed which are the characteristic variables of the stability problem, namely,

$$\frac{\beta_1 \delta}{U_m} = \text{dimensionless amplification quantity}$$

$$\frac{\lambda}{\delta} = \text{dimensionless disturbance wave length}$$

$$\frac{U_m \delta}{\nu} = R = \text{Reynolds number}$$

$$\frac{\frac{g}{\delta} \frac{\Delta \rho}{\rho}}{\left(\frac{U_m}{\delta}\right)^2} = \Theta = \text{Richardson number}$$

$$\frac{\Delta \rho}{\rho} = \text{corresponding density variation} \left(\text{or } \frac{U_m^2}{\delta g} = \frac{\Delta \rho}{\rho} / \Theta = \text{Froude number} \right).$$

The most general solution of the stability problem therefore implies the finding of the function

$$G\left(\frac{\beta_1 \delta}{U_m}, \frac{\lambda}{\delta}, R, \Theta, \frac{\Delta \rho}{\rho}\right) = 0 \quad (3)$$

In an earlier report (reference 22), this very general function G of five variables had been specialized to the effect that the amplification for a homogeneous fluid ($\Theta = 0, \Delta \rho / \rho = 0$) was computed as function of the Reynolds number and the wave length of the disturbance, hence

$$G_1\left(\frac{\beta_1 \delta}{U_m}, \frac{\lambda}{\delta}, R\right) = 0$$

The subsequent investigations are based upon the limitation employed in almost all other stability studies, that only wave lengths of disturbance $\lambda = \lambda_0$ which are neither damped nor amplified are involved (neutral disturbances).

Hence, it is assumed that $\beta_1 = 0$ and the more special function

$$G_2\left(\frac{\lambda_0}{\delta}, R, \Theta, \frac{\Delta \rho}{\rho}\right) = 0 \quad (4)$$

which depends on only four variables is determined in place of (3). In contrast with the homogeneous fluid there results at once through the introduction of the density stratification two new nondimensionals, namely Θ and $\Delta \rho / \rho$, where Θ gives the effect on the potential energy of gravity of the density stratification and $\Delta \rho / \rho$ represents the effect of the stratification of densities on the inertia. The ratio of these two

nondimensional, that is, $\frac{1}{\Theta} \frac{\Delta \rho}{\rho} = \frac{U_m^2}{g\delta} = F^2$, is the square of a Froude number. In place of Θ and $\Delta \rho/\rho$, Θ and F can be chosen as independent variables, in which case

$$G_2\left(\frac{\lambda_0}{\delta}, R, \Theta, F\right) = 0 \quad (4a)$$

replaces (4).

The simultaneous addition of two new independent variables resulting from the stratification means an unusual increase in the calculation, which in the homogeneous case is already quite extensive. This function G_2 is calculated explicitly numerically but not analytically. To arrive at a relation that is conveniently comparable with experiment, the disturbance wave length λ_0 is, ultimately, eliminated, by searching for that Reynolds number $R = R_k$ (critical Reynolds number) where precisely a single undamped disturbance still exists, while all others are damped.

$$G_3(R_k, \Theta, F) = 0$$

or

$$R_k = R_k(\Theta, F) \quad (6)$$

is then also determined and from it a critical Richardson number Θ_k of the profile, that is, the maximum value of Θ at which an undamped disturbance wave length still exists at all. This depends therefore also on the Froude number F : $\Theta_k = \Theta_k(F)$. According to the energy considerations by Prandtl and Richardson, $\Theta_k = 1$ is independent of the velocity profile; according to Taylor and Goldstein, who disregarded the inertia effect of the stratification with respect to the gravity effect, that is, assumed $F = 0$, $\Theta_k = 1/4$ for certain linear velocity profiles and uniform density distributions.

Tollmien's value of the critical Reynolds number for plate flow in homogeneous fluid was $R_k = (U_m \delta^* / \nu)_k = 575$, hence $R_k(\Theta, F) = 575$ for $\Theta = 0$ and all F . It is to be expected that $(\Theta > 0)R_k(\Theta) > 575$ for stable stratification, and $(\Theta < 0)R_k(\Theta) < 575$ for unstable stratification.

The stability study with stratification of densities is, first, made rather general for a profile with curvature different from zero and then the laminar flow along a flat plate is calculated as example, so as to tie in with Tollmien and with experimental data. One more important assumption is made, namely, that the stratification exists only in the boundary layer, while outside of it, where the velocity of the laminar flow is constant, the density itself is assumed to be constant. For the distribution of the density, an exponential law

$$\rho_0(y) = \rho_{0w} e^{-\gamma y}$$

hence that $\frac{1}{\rho} \frac{d\rho}{dy} = -\gamma$ is a constant, is assumed later for reasons of mathematical simplification.

2. THE GENERAL EQUATION OF DISTURBANCE

Suppose the undisturbed plane laminar flow has the direction of the horizontal x axis and is given solely as function of the height y : $U = U(y)$ (y axis at right angles, upward). The density of the undisturbed flow at height y is unknown: $\rho_0 = \rho_0(y)$. Limited to the two-dimensional case, the equations of motion and of continuity read

$$\left. \begin{aligned} \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) &= -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \\ \rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) &= -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - \rho g \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{aligned} \right\} \quad (7a)$$

The presumed incompressibility is expressed by the fact that every particle maintains its density during the motion, hence

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} = 0 \quad (7b)$$

The plane disturbance flow is assumed as a wave motion advancing in x direction¹.

We put therefore

$$\left. \begin{aligned} u &= U(y) + u' & v &= v' \\ \text{where} & & & \\ u' &= u_1(y)e^{i(\alpha x - \beta t)} & v' &= v_1(y)e^{i(\alpha x - \beta t)} \end{aligned} \right\} \quad (8a)$$

are the components of the disturbance motion; $\alpha = 2\pi/\lambda$ is real and denotes the spatial angular frequency; β is, in general, complex; $\beta = \beta_r + i\beta_i$. β_r is the angular frequency of the disturbance motion with respect to time; β_i indicates the amplification or damping, depending upon whether positive or negative.

The pressure p and the density ρ are expressed as

$$\left. \begin{aligned} p &= p_0(y) + p' & \rho &= \rho_0(y) + \rho' \\ \text{with} & & & \\ p' &= p_1(y)e^{i(\alpha x - \beta t)} & \rho' &= \rho_1(y)e^{i(\alpha x - \beta t)} \end{aligned} \right\} \quad (8b)$$

¹This implies no limitation for the general character of the stability study, since Squire (Proc. Roy. Soc. A. vol. 142, 1933) indicated that when a plane flow is unstable against three-dimensional disturbances at a certain Reynolds number, it is unstable at an even lower Reynolds number for two-dimensional disturbances. The two-dimensional disturbances are therefore more "dangerous" than the three-dimensional.

Inserting (8a), (8b), in (7a), (7b) gives

$$\left. \begin{aligned} \rho_0 \left[i(\alpha U - \beta) u_1 + U' v_1 \right] &= -i\alpha p_1 + \mu(-\alpha^2 u_1 + u_1'') \\ \rho_0 \left[i(\alpha U - \beta) v_1 + g \rho_1 \right] &= -p_1' + \mu(-\alpha^2 v_1 + v_1'') \\ v_1' + i\alpha u_1 &= 0 \\ i(\alpha U - \beta) \rho_1 + v_1 \rho_0' &= 0 \end{aligned} \right\} \quad (9)$$

only terms of the first order being maintained; the dash denotes the differentiation with respect to y . There are four equations for the four unknowns u_1 , v_1 , p_1 , and ρ_1 , which, after elimination of u_1 , p_1 , and ρ_1 leaves one equation for v_1 namely

$$\begin{aligned} (\alpha U - \beta)^2 (v_1''' - \alpha^2 v_1) - \alpha(\alpha U - \beta) U'' v_1' + \frac{\rho_0'}{\rho_0} \left\{ -g\alpha^2 v_1 \right. \\ \left. + (\alpha U - \beta)^2 v_1' - \alpha(\alpha U - \beta) U' v_1 \right\} = -i v (\alpha U - \beta) \left\{ v_1'''' - 2\alpha^2 v_1'' \right. \\ \left. + \alpha^4 v_1 + \frac{\rho_0'}{\rho_0} (v_1''' - \alpha^2 v_1') \right\} \end{aligned} \quad (10)$$

$v = \mu/\rho_0$ is assumed constant².

A flow function of the disturbance motion is introduced

$$\Psi(x, y, t) = \varphi(y) e^{i(\alpha x - \beta t)}$$

²According to it, the viscosity μ itself varies with y , and in such a way that $\frac{\mu(y)}{\rho_0(y)} = \text{const.}$ But the terms with $\frac{\partial \mu}{\partial y}$ are disregarded as small of higher order.

so that, as a result of $u' = \frac{\partial \psi}{\partial y}$ and $v' = -\frac{\partial \psi}{\partial x}$

$$u_1 = \phi'(y) \quad v_1 = -i\alpha\phi(y)$$

Putting further $c = \beta/\alpha$, where the real part of c signifies the phase velocity of the disturbance motion, and introducing nondimensional variables by referring all velocities to the maximum velocity U_m of the laminar flow and all lengths to the boundary-layer thickness δ of the laminar flow, the fundamental differential equation for the flow function ϕ of the disturbance motion follows by (10) as

$$\begin{aligned} (U - c)^2(\phi'' - \alpha^2\phi) - (U - c)U''\phi - \frac{\delta}{\rho_0} \frac{d\rho_0}{dy} \frac{g\delta}{U_m^2} \phi \\ + \frac{\delta}{\rho_0} \frac{d\rho_0}{dy} \left[(U - c)^2\phi' - (U - c)U'\phi \right] = -\frac{1}{\alpha R}(U - c) \left[\phi'''' - 2\alpha^2\phi'' \right. \\ \left. + \alpha^4\phi + \frac{\delta}{\rho_0} \frac{d\rho_0}{dy} (\phi'''' - \alpha^2\phi') \right] \end{aligned} \quad (11)$$

The dash indicates the differentiation with respect to $\frac{y}{\delta}$; U stands

for $\frac{U}{U_m}$, c for $\frac{c}{U_m}$, α for $\alpha\delta$, and $R = \frac{U_m\delta}{\nu}$ is the Reynolds number.

The boundary conditions are: disappearance of both disturbance components for $y = 0$ and $y = \infty$, when a laminar flow is assumed that at $y = 0$ is bounded by a fixed wall, while being infinite upward; hence $\phi = \phi' = 0$ for $y = 0$ and $y = \infty$. With these boundary conditions, the stability problem proves to be a characteristic value problem

of the kind that each specified group of four values $\alpha, R, \frac{\delta}{\rho_0} \frac{d\rho_0}{dy} \frac{g\delta}{U_m^2},$

$\frac{\delta}{\rho_0} \frac{d\rho_0}{dy}$ gives a complex value of c , whose imaginary part decides between

amplification or damping of the particular disturbance in the particular flow, which is characterized by a Reynolds, Richardson, and Froude number. However, the study is restricted, as stated in the introduction, to disturbances lying between stability and instability, hence for which c is purely real. For this case, $c < U_m$ and $\alpha \ll R$, provided that U'' does not change signs in the laminar flow. The laminar flow contains, therefore, a layer in which the phase velocity of the disturbance motion is equal to the primary flow velocity. This layer is called critical layer ($y = y_{crit}$) and plays a prominent part in the subsequent study.

3. THE FRICTIONLESS DISTURBANCE EQUATION

Because αR is very great at the stability limit, information about the solutions of the general disturbance equation (11) can be obtained by analyzing the frictionless disturbance differential equation

$$\begin{aligned} (U - c)^2(\varphi'' - \alpha^2\varphi) - (U - c)U''\varphi - \frac{\delta}{\rho_0} \frac{d\rho_0}{dy} \frac{g\delta}{U_m^2}\varphi \\ + \frac{\delta}{\rho_0} \frac{d\rho_0}{dy} \left[(U - c)^2\varphi' - (U - c)U'\varphi \right] = 0 \end{aligned} \quad (12)$$

Visualize the speed of the laminar flow near the critical layer where $U = c$ expanded in a series

$$U - c = U_k'(y - y_k) + \frac{1}{2} U_k''(y - y_k)^2$$

and broken off with the quadratic term. By (12)

$$\begin{aligned}
 & \left[(y - y_k) + \frac{1}{2} \frac{U_k''}{U_k'} (y - y_k)^2 \right]^2 (\phi'' - \alpha^2 \phi) - \frac{U_k''}{U_k'} (y - y_k) \left[1 + \frac{1}{2} \frac{U_k''}{U_k'} (y - y_k) \right] \phi \\
 & + K\phi + L \left[(y - y_k) + \frac{1}{2} \frac{U_k''}{U_k'} (y - y_k)^2 \right] \left\{ \left[1 + \frac{U_k''}{U_k'} (y - y_k) \right] \phi \right. \\
 & \left. - \left[(y - y_k) + \frac{1}{2} \frac{U_k''}{U_k'} (y - y_k)^2 \right]^2 \phi \right\} = 0
 \end{aligned} \tag{13}$$

where

$$\left. \begin{aligned} K &= -\frac{\delta}{\rho_0} \frac{d\rho_0}{dy} \frac{g\delta}{U_m^2} \frac{1}{U_k'^2} = -\frac{g}{\rho_0} \frac{d\rho_0}{dy} / \left(\frac{dU}{dy} \right)_k^2 \\ L &= -\frac{\delta}{\rho_0} \frac{d\rho_0}{dy} \end{aligned} \right\} \tag{14a}$$

To make K and L constant in the boundary layer, the simple exponential distribution law

$$\rho_0 = \rho_0 e^{-\gamma y} \tag{15}$$

is assumed for the density. Then

$$\left. \begin{aligned} K &= \frac{g\gamma}{\left(\frac{dU}{dy} \right)_k^2} \\ L &= \delta\gamma \end{aligned} \right\} \tag{14b}$$

The frictionless differential equation contains the two nondimensional K and L as a result of the stratification of densities, K representing the gravity effect and L the inertia effect of the stratification on the disturbance motion. Taylor and Goldstein ignored the inertia effect with respect to the gravity effect almost altogether which, according to (13), seems to be permissible only when $L \ll K$, or

according to (14b) when $\left(\frac{dU}{dy}\right)_k^2 \frac{\delta}{g} \approx \frac{U_m^2}{\delta g} \ll 1$, that is, when the Froude

number of the flow is small with respect to unity. But, from the subsequent calculations, it will be clear that at equal numerical value of K and L the influence of the gravity effect on the stability is far greater than that of the inertia effect. For the measurements with

which the calculation is to be compared, $\frac{U_m^2}{\delta g}$ even exceeds unity, hence

$L > K$. Since it is impossible to give an estimate of the amount of the inertia effect involved, the complete calculation is carried out with gravity and inertia effect.

The next step is the integration of (13). The solution of this differential equation is visualized as expansion in powers of K and L

$$\varphi = \varphi^{(0)} + K\varphi^{(K)} + L\varphi^{(L)} + \dots \quad (16)$$

broken off with the linear terms in K and L . It implies, proceeding from the homogeneous fluid, the study of a slightly stratified flow

($K = L = 0$ is the homogeneous solution); $\varphi^{(0)}$ is the already known solution of the differential equation of disturbance of the homogeneous fluid and $\varphi^{(K)}$ and $\varphi^{(L)}$ are additional solutions due to stratification. This expansion, and particularly the stopping with the linear terms, is permissible only for values of K and L which are small with respect to unity.

As the laminar flow, whose stability is to be studied, is approximated by a constant, a linear, and a quadratic function, we integrate in the following equation (13) for constant, linear, and quadratic velocity distribution. The calculations for $\varphi^{(K)}$ and $\varphi^{(L)}$ run parallel.

For the zone of constant laminar-flow velocity, the density is assumed constant, as stated in the introduction. The disturbance differential equation then is

$$\varphi'' - \alpha^2 \varphi = 0$$

with the decaying solution

$$\varphi = e^{-\alpha y} \quad (17)$$

for great y .

For linear velocity distribution with the new variable, it is

$$\left. \begin{aligned} \frac{y - y_k}{y_k} &= y_1 \\ \alpha y_k &= \alpha_1 \end{aligned} \right\} \quad (18)$$

and with

by (13)

$$y_1^2 \left(\frac{d^2 \varphi}{dy_1^2} - \alpha_1^2 \varphi \right) + K\varphi + Ly_k \left(y_1 \varphi - y_1^2 \frac{d\varphi}{dy_1} \right) = 0 \quad (19)$$

For the homogeneous fluid, there follows from

$$\frac{d^2 \varphi(0)}{dy_1^2} - \alpha_1^2 \varphi(0) = 0$$

the solutions

$$\varphi_1(0) = \frac{\sinh \alpha_1 y_1}{\alpha} \quad \varphi_2(0) = \cosh \alpha_1 y_1 \quad (20)$$

By (16) and (19) the linear inhomogeneous differential equation for the additional solution for $\varphi^{(K)}$ is

$$y_1^2 \left(\frac{d^2 \varphi^{(K)}}{dy_1^2} - \alpha_1^2 \varphi^{(K)} \right) = -\varphi(0) \quad (21a)$$

and for the additional solution $\varphi^{(L)}$

$$y_1 \left(\frac{d^2 \varphi^{(L)}}{dy_1^2} - \alpha_1^2 \varphi^{(L)} \right) = -y_k \left(\varphi^{(0)} - y_1 \frac{d\varphi^{(0)}}{dy_1} \right) \quad (21b)$$

the homogeneous part of both equations being the same as the differential equation for $\varphi^{(0)}$. Hence, only a particular integral of (21a) and (21b) has to be determined to produce the general integrals $\varphi^{(K)}$ and $\varphi^{(L)}$. These integrals can, however, be determined immediately by integration. The result is a fundamental system for the solutions $\varphi^{(K)}$ and $\varphi^{(L)}$ by 21a, 21b) in the form

$$\left. \begin{aligned} \varphi_v^{(K)} &= + \varphi_v^{(0)} - \frac{\varphi_1^{(0)}}{y_k} \int_{y_0}^{y_1} \frac{\varphi_2^{(0)} \varphi_v^{(0)}}{y_1^2} dy_1 + \frac{\varphi_2^{(0)}}{y_k} \int_{y_0}^{y_1} \frac{\varphi_1^{(0)} \varphi_v^{(0)}}{y_1^2} dy_1 \\ \varphi_v^{(L)} &= + \varphi_v^{(0)} - \varphi_1^{(0)} \int_{y_0}^{y_1} \varphi_2^{(0)} \left(\frac{\varphi_v^{(0)}}{y_1} - \frac{d\varphi_v^{(0)}}{dy_1} \right) dy_1 \\ &\quad + \varphi_2^{(0)} \int_{y_0}^{y_1} \varphi_1^{(0)} \left(\frac{\varphi_v^{(0)}}{y_1} - \frac{d\varphi_v^{(0)}}{dy_1} \right) dy_1 \quad (v = 1, 2) \end{aligned} \right\} \quad (22)$$

The choice of the lower integration limit is immaterial; the present choice is the point $y = y_0$, where the linear velocity profile is joined to the parabolic.

For parabolic velocity distribution, which is written in the form

$$U = (a - y)^2 \quad (23)$$

($a = \text{given constant}$), there results from the new variables

$$y_2 = \frac{y - y_k}{a - y_k}$$

and

$$\alpha_2 = (\alpha - y_k)\alpha$$

from equation (13) the disturbance differential equation

$$y_2^2 \left(1 - \frac{y_2}{2}\right)^2 \left(\frac{d^2\varphi}{dy_2^2} - \alpha_2^2 \varphi\right) + y_2 \left(1 - \frac{y_2}{2}\right) \varphi + K\varphi$$

$$+ L(a - y_k) \left[y_2 \left(1 - \frac{y_2}{2}\right) (1 - y_2) \varphi - y_2^2 \left(1 - \frac{y_2}{2}\right)^2 \frac{d\varphi}{dy_2} \right] = 0 \quad (24)$$

The solution $\varphi(0)$ of this equation for the homogeneous fluid ($K = L = 0$) has already been computed by Tollmien and others. A fundamental system is given by

$$\frac{\varphi_1(0)}{a - y_k} = d_1 y_2 + d_2 y_2^2 + d_3 y_2^3 + \dots$$

$$\varphi_2(0) = e_0 + e_1 y_2 + e_2 y_2^2 + \dots - \frac{\varphi_1(0)}{a - y_k} \log y_2,$$

$$\text{where } d_1 = 1; \quad d_2 = -\frac{1}{2}; \quad d_3 = \frac{\alpha_2^2}{6}; \quad d_4 = \frac{\alpha_2^2}{18};$$

(25)

$$d_n = \frac{1}{2n(n-1)} \left[n(n-3)d_{n-1} + 2\alpha_2^2 d_{n-2} - \alpha_2^2 d_{n-3} \right] \quad n = 2, 3, \dots$$

$$e_0 = 1; \quad e_1 = 0; \quad e_2 = \frac{\alpha_2^2}{2} - 1; \quad e_3 = \frac{1}{8} + \frac{\alpha_2^2}{18}; \quad \dots$$

$$e_n = \frac{1}{2n(n-1)} \left[n(n-3)e_{n-1} + 2\alpha_2^2 e_{n-2} - \alpha_2^2 e_{n-3} - (2n-3)d_{n-1} + 2(2n-1)d_n \right]$$

$$n = 2, 3, \dots$$

By (16) and (24), the differential equations for the additional solutions $\varphi(K)$ and $\varphi(L)$ are

$$y_2^2 \left(1 - \frac{y_2}{2}\right)^2 \left(\frac{d^2 \varphi(K)}{dy_2^2} - \alpha_2^2 \varphi(K) \right) + y_2 \left(1 - \frac{y_2}{2}\right) \varphi(K) = -\varphi(0)$$

and

$$y_2 \left(1 - \frac{y_2}{2}\right) \left(\frac{d^2 \varphi(L)}{dy_2^2} - \alpha_2^2 \varphi(L) \right) + \varphi(L) = -(a - y_k) \left[(1 - y_2) \varphi(0) - y_2 \left(1 - \frac{y_2}{2}\right) \frac{d\varphi(0)}{dy_2} \right]$$

The homogeneous part of this inhomogeneous differential equation for $\varphi(K)$ and $\varphi(L)$ again agrees with the differential equation for $\varphi(0)$; hence a fundamental system for the general solutions $\varphi(K)$ and $\varphi(L)$ is obtained again by quadrature:

$$\left. \begin{aligned} \varphi_v(K) &= \varphi_v(0) - \frac{\varphi_1(0)}{a - y_k} \int_{y_1}^{y_2} \frac{\varphi_2(0) \varphi_v(0)}{y_2^2 \left(1 - \frac{y_2}{2}\right)^2} dy_2 \\ &+ \frac{\varphi_2(0)}{a - y_k} \int_{y_0}^{y_2} \frac{\varphi_1(0) \varphi_v(0)}{y_2^2 \left(1 - \frac{y_2}{2}\right)^2} dy_2 \quad (v = 1, 2) \\ \varphi_v(L) &= \varphi_v(0) - \varphi_1(0) \int_{y_0}^{y_2} \varphi_2(0) \left[\frac{(1 - y_2) \varphi_v(0)}{y_2 \left(1 - \frac{y_2}{2}\right)} - \varphi_v(0)' \right] dy_2 \\ &+ \varphi_2(0) \int_{y_0}^{y_2} \varphi_1(0) \left[\frac{(1 - y_2) \varphi_v(0)}{y_2 \left(1 - \frac{y_2}{2}\right)} - \varphi_v(0)' \right] dy_2 \quad (v = 1, 2) \end{aligned} \right\} \quad (26)$$

Herewith the frictionless solutions of the stratified fluid for linear and parabolic velocity distributions are obtained. Of these integrals only $\varphi_1^{(0)}$ in the singular point $y = y_k$ of the differential equation is regular, while $\varphi_2^{(0)}$, $\varphi_1^{(K)}$, and $\varphi_2^{(K)}$, $\varphi_1^{(L)}$, and $\varphi_2^{(L)}$ exhibit singularities, where either they themselves or their derivatives contain terms with $\log(y - y_k)$ or $(y - y_k)^{-1}$. These singularities are obviously attributable to the disregarded friction, and disappear as soon as they are properly allowed for. The integration along the real y axis must be replaced near the singular point $y = y_k$ by the integration in the complex along a semicircle around point $y = y_k$. First, the sense of rotation about point $y = y_k$ must be decided upon, that is, the choice of the branch of the log at transition from positive to negative $y - y_k$. Tollmien indicated this transitional substitution for $\varphi_2^{(0)}$ which is obtainable when the entire equation (11) is discussed in close vicinity of the critical point $y = y_k$.

4. BEHAVIOR OF THE SOLUTIONS IN THE NEIGHBORHOOD OF

THE CRITICAL LAYER

To this end, a small zone around point $y = y_k$ (transitional zone) is assumed, in which the substitution of $U - c$ by $U_k'(y - y_k)$ and U' and U'' by U_k' and U_k'' is sufficiently accurate. In addition

$$y - y_k = (\alpha R U_k')^{-1/3} \eta = \epsilon \eta \quad (27)$$

where, on account of the small value of ϵ , even in the transitional zone (that is, small values of $y - y_k$), η can assume great values.

By (11) and (27), the differential equation for $\varphi(\eta)$ is

$$\begin{aligned} i\eta\varphi'''' + \eta\varphi''(\eta - 2i\alpha\epsilon^2) - \eta\varphi\left(\epsilon\frac{U_k''}{U_k'} + \epsilon^2\alpha^2\eta - i\epsilon^4\alpha^4\right) \\ + K\varphi + \frac{L}{U_k'^2}\epsilon\eta(\varphi - \eta\varphi') = 0 \end{aligned} \quad (28)$$

where the terms with $\frac{1}{R} \frac{\delta}{\rho_0} \frac{d\rho_0}{dy}$ in (11) are disregarded as small of the second order. Tollmien (reference 13) demonstrated by means of the corresponding differential equation for homogeneous fluid ($K = L = 0$) that the transitional substitution for $\varphi_2^{(0)}$ reads as follows

$$\begin{aligned} y - y_k > 0: \varphi_2^{(0)} &= e_0 + e_1 y + e_2 y^2 + \dots + \frac{U_k''}{U_k} \varphi_1^{(0)} \log(y - y_k) \\ y - y_k < 0: \varphi_2^{(0)} &= e_0 + e_1 y + e_2 y^2 + \dots + \frac{U_k''}{U_k} \varphi_1^{(0)} (\log|y - y_k| - i\pi) \end{aligned} \quad (29)$$

Thus, of the infinitely many values of $\log(y - y_k) + 2k\pi i$ ($k =$ positive or negative whole number), the one with $k = -1$ proves physically real. A calculation similar to that made by Tollmien for $\varphi_2^{(0)}$ gives the same result for $\varphi_1^{(K)}$, $\varphi_2^{(K)}$, and $\varphi_2^{(L)}$, namely, that the terms with $\log(y - y_k)$ for negative $y = y_k$ must be replaced by $\log|y - y_k| - i\pi$. A further approximate calculation for the homogeneous fluid, which in (28) includes only the highest friction term ($i\varphi''' + \eta\varphi'' = 0$), gives the two other solutions $\varphi_3^{(0)}$ and $\varphi_4^{(0)}$ needed for representing the general integral of the differential equation of disturbance of the fourth order (11). It is

$$\varphi_{3,4}^{(0)} = \int_{-\infty}^{\eta} d\eta \int_{-\infty}^{\eta} \eta^{1/2} H_{1/3}^{(1),(2)} \left[\frac{2}{3}(i\eta^{3/2}) \right] d\eta \quad (30)$$

$H^{(1),(2)}$ is the Hankel function of the first and second kind, respectively.

Additional solutions for φ_3 and φ_4 , namely,

$$\varphi_{3,4} = \varphi_{3,4}^{(0)} + K\varphi_{3,4}^{(K)} + L\varphi_{3,4}^{(L)} + \dots$$

should also be computed, but the subsequent paragraphs indicate that the calculation of $\varphi_{3,4}^{(K)}$ and $\varphi_{3,4}^{(L)}$ is superfluous because

$$\frac{\varphi_{3,4}^{(K)}}{\varphi_{3,4}^{(0)}} \ll \frac{\varphi_{1,2}^{(K)}}{\varphi_{1,2}^{(0)}} \quad \text{and} \quad \frac{\varphi_{3,4}^{(L)}}{\varphi_{3,4}^{(0)}} \ll \frac{\varphi_{1,2}^{(L)}}{\varphi_{1,2}^{(0)}} \quad (31)$$

The effect of the stratification of densities on the frictionless solutions $\varphi_{1,2}$ is, therefore, substantially greater than on the friction solutions $\varphi_{3,4}$, so the latter can be disregarded in the approximation; this simplifies the calculation considerably. The result is analagous to the conditions in a flow with stratification of centrifugal forces. In the earlier Göttingen report (reference 20), it also had been shown that in stable stratification due to centrifugal forces the effect of this stratification on the frictionless solutions $\varphi_{1,2}$ is very much greater than on the friction solutions $\varphi_{3,4}$.

The following proof is given:

5. THE VARYING EFFECT OF THE STRATIFICATION OF DENSITIES ON THE TWO PAIRS OF SOLUTIONS

Desired is a representation of the four integrals $\varphi_1, \varphi_2, \varphi_3$, and φ_4 for the wall proximity (very small y) on the specific assumption that the phase velocity c of the disturbance motion is very much higher than the velocity near the wall. In other words, the critical point is to lie outside the layer where the solutions are analyzed. For the wall proximity $U - c$ can then be replaced by $-c$ and U'' by U_w'' .

On account of $-\frac{1}{\rho_0} \frac{d\rho_0}{dy} = \gamma$, according to equation (15), the general equation of disturbance for a layer near the wall is by equation (11)

$$\varphi''' + (i\alpha R c - 2\alpha^2)\varphi'' + \left[\alpha^4 + i\alpha R \left(U_w'' + \frac{\gamma g}{c} - \alpha^2 c \right) \right] \varphi = 0$$

The solutions of this equation with constant coefficients are of the form

$$\varphi_v = e^{k_v y} \quad v = 1, 2, 3, 4$$

k_v are the four roots of the equation of the fourth degree

$$k^4 + (i\alpha Rc - 2\alpha^2)k^2 + \left[\alpha^4 + i\alpha R \left(U_w'' + \frac{\gamma g}{c} - \alpha^2 c \right) \right] = 0$$

Both pairs of solutions are given by

$$k^2 = -\frac{1}{2}(i\alpha Rc - 2\alpha^2) \pm \sqrt{\frac{1}{4}(i\alpha Rc - 2\alpha^2)^2 - \alpha^4 - i\alpha R \left(U_w'' + \frac{\gamma g}{c} - \alpha^2 c \right)}$$

hence

$$k_{1,2}^2 = \alpha^2 - \frac{1}{c} \left(U_w'' + \frac{\gamma g}{c} \right)$$

or when $U_w'' = 0$

$$k_{1,2}^2 = \alpha^2 \left[1 - K \left(\frac{U_w'}{\alpha c} \right)^2 \right], \text{ since } K = \frac{\gamma g}{U_w'^2}$$

or

$$k_{1,2} = \pm \alpha \left[1 - \frac{K}{2} \left(\frac{U_w'}{\alpha c} \right)^2 \right]$$

Likewise, when α^2 is disregarded relative to αRc

$$k_{3,4}^2 = -i\alpha Rc + K \left(\frac{U_w'}{c} \right)^2$$

or

$$k_{3,4} = \pm \sqrt{i\alpha Rc} \left[1 - \frac{K i \alpha}{2 Rc} \left(\frac{U_w'}{\alpha c} \right)^2 \right]$$

From

$$\phi_1 = \phi_1(0) + K\phi_1(K) = e^{\alpha \left[1 - \frac{K}{2} \left(\frac{U_w'}{\alpha c} \right)^2 \right] y}$$

follows

$$\frac{\varphi_1(K)}{\varphi_1(0)} = - \frac{\alpha y}{2} \left(\frac{U_w'}{\alpha c} \right)^2$$

and likewise from

$$\varphi_3 = \varphi_3^{(0)} + K\varphi_3^{(K)} = e^{-\sqrt{1\alpha Rc} \left[1 - \frac{K}{2} \left(\frac{\alpha}{Rc} \right) \left(\frac{U_w'}{\alpha c} \right) \right] y}$$

$$\frac{\varphi_3(K)}{\varphi_3(0)} = + \frac{\alpha y}{2} \left(\frac{U_w'}{\alpha c} \right)^2 \sqrt{\frac{1\alpha}{Rc}}$$

Hence

$$\left| \frac{\varphi_3(K)}{\varphi_1(K)} \times \frac{\varphi_1(0)}{\varphi_3(0)} \right| = \sqrt{\frac{\alpha}{Rc}} \quad (32)$$

With δ as reference length, the flow along the flat plate is, at the stability limit, (reference 22) approximately

$$\alpha = 0.84; \quad R = 1700; \quad c = 0.42$$

hence

$$\left| \frac{\varphi_3(K)}{\varphi_1(K)} \times \frac{\varphi_1(0)}{\varphi_3(0)} \right| = \sqrt{\frac{1}{850}} \approx \frac{1}{29} \ll 1$$

The same holds true for $\varphi_2^{(K)}$ and $\varphi_4^{(K)}$, and a similar derivation can be given for $\varphi_3^{(L)}$ and $\varphi_4^{(L)}$, with which equation (31) is therefore proved.

This indicates that the effect of the stratification of densities on the slowly variable solutions φ_1 and φ_2 , which are as $e^{\pm \alpha y}$, is much greater than on the rapidly varying solutions which are as $e^{\pm \sqrt{\alpha R} cy}$.

6. FORMULATION OF THE CHARACTERISTIC VALUE PROBLEM

The next problem deals with the boundary conditions and also the formulation of the characteristic value problem. A basic velocity profile is assumed which increases from value zero to a maximum value U_m which it then maintains constant. By (17), the solution of the differential equation of disturbance for the zone of constant velocity, where constant density is assumed also, is: $\varphi = e^{-\alpha y}$. With $y = a$ denoting the point of contact with the zone of constant U , the boundary condition at this point reads

$$\varphi_a' + \alpha \varphi_a = 0 \quad (33)$$

The general integral reads

$$\varphi = C_1 \varphi_1 + C_2 \varphi_2 + C_3 \varphi_3 + C_4 \varphi_4$$

The boundary conditions ($\varphi = \varphi' = 0$ for $y = 0$ and $y = \infty$) can be considerably simplified on the basis of the particular properties of the four integrals φ_1 , φ_2 , φ_3 , and φ_4 . Thus, the integral φ_4 ,

which for very great y increases infinitely (like $e^{+\sqrt{\alpha R} cy}$) need not be considered in the general integral, so that $C_4 = 0$. Furthermore,

the integral φ_3 , which is as $e^{-\sqrt{\alpha R} cy}$, needs to be considered only at the wall, while in the connecting point on the zone with constant velocity ($y = a$) it can be regarded as zero with good approximation.

For $y = a$, the boundary conditions read, by (33) therefore, simply

$$C_1 \varphi'_{1a} + C_2 \varphi'_{2a} + \alpha (C_1 \varphi_{1a} + C_2 \varphi_{2a}) = 0$$

or, when putting

$$\varphi'_{va} + \alpha \varphi_{va} = \Phi_{va} \quad v = 1, 2 \quad (33a)$$

$$C_1 \phi_{1a} + C_2 \phi_{2a} = 0 \quad (34a)$$

The boundary condition at the wall ($y = 0$) reads

$$C_1 \phi_{1w} + C_2 \phi_{2w} + C_3 \phi_{3w} = 0 \quad (34b)$$

$$C_1 \phi'_{1w} + C_2 \phi'_{2w} + C_3 \phi'_{3w} = 0 \quad (34c)$$

hence the three homogeneous equations (34a), (34b), and (34c) as boundary conditions. If one solution ϕ other than zero exists, the determinant

$$\begin{vmatrix} \phi_{1a} & \phi_{2a} & 0 \\ \phi_{1w} & \phi_{2w} & \phi_{3w} \\ \phi'_{1w} & \phi'_{2w} & \phi'_{3w} \end{vmatrix} = 0 \quad (35)$$

must disappear, wherewith the stability investigation as a characteristic value problem of the differential equation of disturbance is proved. The complex equation (35) depends, besides the constants of the profile used as basis, upon the parameters α , c , R , K , and L - all of which, especially c , are purely real - since only the transitional points from stability to instability are involved. The functions ϕ_{vw} and ϕ_{va} are usually complex (compare equation (29) and (30)). Visualizing the parameter c eliminated from the two real equations with which the complex equation (35) is equivalent leaves one equation in which α , R , K , and L are contained. This equation gives, for fixed K and L , the indifference curve in the αR plane corresponding to this stratification (K , L), and which separates the damped from the undamped disturbances. Thus the solution of the stability problem terminates in the discussion of equation (35).

Calculation of the determinant (35) gives

$$\frac{\phi_{3w}}{\phi'_{3w}} = \frac{\phi_{2w}\phi_{1a} - \phi_{1w}\phi_{2a}}{\phi'_{2w}\phi_{1a} - \phi'_{1w}\phi_{2a}} \quad (36)$$

This equation is then expanded in powers of the stratification quantities K and L corresponding to the fact that, starting from the homogeneous fluid, the stability study for slight stratification is to be carried out. Only the right-hand side of the equation has to be expanded, which contains the frictionless solutions exclusively, hence is not dependent on the Reynolds number R . On the other hand, the solution ϕ_3 is not dependent on the laminar flow serving as basis. It can be summarily taken over from Tietjens' report. The left-hand side of equation (36) depends only on

$$\eta_w = -y_k(\alpha R U'_k)^{1/3} = -\frac{y_k}{\epsilon} \quad (37)$$

For $D(\eta_w) = -\frac{1}{\epsilon} \frac{\phi_{3w}}{\phi'_{3w}}$, the following table I is obtained according to Tietjens:

Table 1: $D(\eta_w)$ as Function of η_w

$-\eta_w$	$D(\eta_w)$	$-\eta_w$	$D(\eta_w)$
0	0.702 - 0.425 i	3.0	1.400 + 0.515 i
0.5	0.785 - 0.411 i	3.5	1.180 + 1.130 i
1.0	0.920 - 0.389 i	4.0	0.460 + 1.250 i
1.5	1.043 - 0.297 i	4.5	-0.0405 + 0.8080 i
2.0	1.206 - 0.147 i	5.0	0.0057 + 0.3645 i
2.5	1.357 + 0.108 i	5.5	0.1913 + 0.2393 i

After putting

$$-\frac{1}{y_k} \frac{\phi_{2w}\phi_{1a} - \phi_{1w}\phi_{2a}}{\phi'_{2w}\phi_{1a} - \phi'_{1w}\phi_{2a}} = E(\alpha, c, K, L) \quad (38)$$

equation (36) gives

$$-\frac{D(\eta_w)}{\eta_w} = E(\alpha, c, K, L) \quad (39)$$

This complex equation forms the starting point for the subsequent calculations, the aim of which is the explicit representation of the relation between the five quantities α , c , R , K , and L given by this equation.

7. SOLUTION OF THE CHARACTERISTIC VALUE PROBLEM

For the numerical treatment of this equation, it is essential that, of the five parameters on the left-hand side, the two stratification quantities K and L do not occur nor the Reynolds number R on the right-hand side.

The solution of this equation, that is, the calculation of the correlated characteristic values α , c , R , K , and L , is first effected analytically so far as the expansion in powers of K and L is involved. The subsequent treatment of the ensuing equation must be made by numerical graphical method, like in the earlier stability studies (references 21, 22). Since the whole calculation in its analytical, as in its numerical-graphical part, is far too extensive to be reproduced here, a brief outline of the line of reasoning must be sufficient:

The expansion of equation (39) in powers of K and L is formally

$$-\frac{D(\eta_w)}{\eta_w} = E^{(0)}(\alpha, c) + K \frac{\partial E}{\partial K} + L \frac{\partial E}{\partial L} + \dots \quad (39a)$$

$E^{(0)}$ denoting the value of equation (38), when inserting for φ_1, φ_2 the solutions for the homogeneous fluid, that is

$$E^{(0)} = -\frac{1}{y_k} \frac{\varphi_{2w}^{(0)} \varphi_{1a}^{(0)} - \varphi_{1w}^{(0)} \varphi_{2a}^{(0)}}{\varphi_{2w}^{(0)} \varphi_{1a}^{(0)} - \varphi_{1w}^{(0)} \varphi_{2a}^{(0)}} \quad (40)$$

while $\frac{\partial E}{\partial K}$ and $\frac{\partial E}{\partial L}$ are obtained when introducing for all terms in (38) their expansion in powers of K and L , that is

$$\left. \begin{aligned} \phi_{vw} &= \phi_{vw}^{(0)} + K\phi_{vw}^{(K)} + L\phi_{vw}^{(L)} + \dots \\ \phi_{va} &= \phi_{va}^{(0)} + K\phi_{va}^{(K)} + L\phi_{va}^{(L)} + \dots \end{aligned} \right\} (v = 1, 2) \quad (41)$$

$\phi_v^{(K)}$, $\phi_v^{(L)}$, and hence $\phi_{va}^{(K)}$ and $\phi_{va}^{(L)}$ are known according to section 3.

The complex equation (39) is equivalent to two real equations. For the left-hand side of equation (39), the decomposition into real and imaginary part is immediately given according to table 1. The right-hand side contains an imaginary component as a result of the transitional substitution of the integral ϕ_2 (equation (29)). Effecting the indicated expansion in powers of K and L explicitly results, after considerable paper work, in the expressions for the real and imaginary part of

$E^{(0)}$, $\frac{\partial E}{\partial K}$, and $\frac{\partial E}{\partial L}$ as function of the two variables α and c . The

numerical values of $E^{(0)}$, $\frac{\partial E}{\partial K}$, $\frac{\partial E}{\partial L}$, for the velocity profile with which

the calculation was made, are indicated in table 2. The further solution of equation (39) is made by graphical method. With fixed c , K , L the left-hand side of equation (39) is plotted with η_w as parameter, and the right-hand side with α as parameter in a polar diagram (fig. 2).

In general, it results in two intersection points of the $\frac{D}{\eta_w}$ curve with

the $E(\alpha, c, K, L)$ curve for which the parameters α_1 , α_2 , and η_{w1} , η_{w2} are found. The corresponding Reynolds number follows then from the equation

$$\eta_w = -y_k(\alpha R U_k')^{1/3}$$

The great advantage of the expansion in powers of K and L is manifested in the numerical calculation by the fact that the relation of the desired transition points from stability to instability of the stratification condition is verified in very simple manner. Computing the quantities $E^{(0)}$, $\frac{\partial E}{\partial K}$, $\frac{\partial E}{\partial L}$ for fixed c , as function of α , immediately yields the curves $E = E(\alpha, c, K, L)$ for the homogeneous flow and for all states of stratification. From the numerical values of $\frac{\partial E}{\partial K}$ and $\frac{\partial E}{\partial L}$ in table 2, it is apparent that the influence of the gravity effect of the stratification is very much greater than the influence of the inertia effect ($\frac{\partial E}{\partial K} \gg \frac{\partial E}{\partial L}$). In the present calculations, small negative values of K and L were also included, that is, slightly unstable stratifications, such as the flow of an evenly tempered air current above a heated plate, which is a little warmer than the air.

TABLE 2.- $R(E)$, $I(E)$, $R\left(\frac{\partial E}{\partial K}\right)$, $I\left(\frac{\partial E}{\partial K}\right)$, $R\left(\frac{\partial E}{\partial L}\right)$, $I\left(\frac{\partial E}{\partial L}\right)$ AS FUNCTION OF α FOR $c = 0.20; 0.25; 0.30; 0.35; 0.375; 0.40; 0.425; 0.45$.

α	$R(E)$	$I(E)$	$R\left(\frac{\partial E}{\partial K}\right)$	$I\left(\frac{\partial E}{\partial K}\right)$	$R\left(\frac{\partial E}{\partial L}\right)$	$I\left(\frac{\partial E}{\partial L}\right)$	$R(E)$	$I(E)$	$R\left(\frac{\partial E}{\partial K}\right)$	$I\left(\frac{\partial E}{\partial K}\right)$	$R\left(\frac{\partial E}{\partial L}\right)$	$I\left(\frac{\partial E}{\partial L}\right)$
$c = 0.20$							$c = 0.25$					
0.20	0.615	0.009	10.00	2.03	0.820	0.021	0.546	0.026	8.64	2.11	0.730	0.010
.30	.396	.024	10.88	3.22	1.41	.050	.350	.050	9.44	2.80	1.17	.000
.40	.132	.047	11.78	4.49	2.10	.095	.141	.090	10.28	3.59	1.49	-.005
.50	-.205	.085	12.84	5.94	3.04	.159	-.165	.155	11.46	4.37	2.18	-.040
.60	-.610	.150	14.06	7.79	4.10	.263	-.513	.273	12.56	5.08	2.89	-.155
.70												
$c = 0.30$							$c = 0.35$					
0.30	0.642	0.026	7.10	1.32	0.490	-0.034	0.730	0.027	5.92	0.850		
.40	.491	.052	7.74	1.77	.690	-.072	.615	.052	6.45	1.10	0.460	-0.110
.50	.333	.091	8.38	2.15	.800	-.135	.480	.090	6.96	1.28	.570	-.192
.60	.128	.157	9.28	2.55	1.100	-.255	.335	.151	7.56	1.47	.740	-.335
.70	-.111	.260	10.26	2.87	1.460	-.465	.165	.245	8.30	1.56	.910	-.565
.80	-.420	.442	11.80	3.84	1.820	-1.09	-.029	.402	9.08	1.52	1.080	-.985
$c = 0.375$							$c = 0.40$					
0.40	0.652	0.050	5.89	0.820	0.369	-0.118	0.690	0.050	5.48	0.586	0.324	-0.123
.50	.542	.088	6.33	.925	.500	-.208	.590	.087	5.76	.650	.430	-.219
.60	.417	.142	6.85	.939	.632	-.350	.484	.139	6.14	.645	.520	-.355
.70	.285	.232	7.32	.966	.742	-.573	.371	.222	6.58	.506	.590	-.570
.80	.133	.369	7.91	.803	.812	-.923	.250	.339	6.91	.346	.595	-.860
.90	-.024	.574	8.40	.340	.870	-1.402	.129	.525	7.26	.056	.452	-1.26
$c = 0.425$							$c = 0.45$					
0.40	0.720	0.049	4.96	0.380	0.270	-0.125	0.754	0.046	4.58	0.170	0.237	-0.127
.50	.637	.085	5.24	.390	.350	-.223	.681	.082	4.84	.140	.288	-.220
.60	.545	.133	5.51	.330	.423	-.355	.605	.126	5.00	.070	.336	-.342
.70	.452	.209	5.94	.190	.440	-.555	.530	.195	5.36	-.050	.324	-.513
.80	.360	.315	6.12	.000	.395	-.795	.456	.298	5.40	-.030	.222	-.725
.90	.266	.475	6.28	-.220	.210	-1.10	.391	.430	5.44	-.490	.014	-.927

8. APPLICATION TO PLATE FLOW

For the stability study with stratification of densities, the laminar flow along a flat plate has been chosen as example for the numerical calculation. Tollmien made the corresponding investigation for the case of homogeneous fluid (reference 13). Blasius originally computed the laminar velocity distribution along the flat plate (reference 23) according to formulas by Prandtl (reference 24). In the present stability study, the modification in the laminar flow due to the stratification of densities can be disregarded because a simple evaluation³ indicates that it is small, when the boundary layer contains only a slight stratification of densities as it was assumed.

³The force in x direction produced by the stratification per unit volume is:

$$(\rho g) \frac{1}{\rho} \frac{d\rho}{dy} \delta \frac{d\delta}{dx} = \rho g \gamma \delta \frac{d\delta}{dx}$$

and the friction per unit area $\mu \frac{dU}{dy}$

or per unit volume $\frac{\mu}{\delta} \frac{dU}{dy} \sim \frac{\mu U_m}{\delta^2}$

Accordingly, the effect of the stratification of densities on the laminar profile can be disregarded when

$$\rho g \gamma \delta \frac{d\delta}{dx} \ll \frac{\mu U_m}{\delta^2}$$

In that case

$$\delta \sim \sqrt{\frac{\nu x}{U_m}} \quad \frac{U_m \delta}{\nu} \sim \sqrt{\frac{U_m x}{\nu}} \quad \frac{d\delta}{dx} \sim \sqrt{\frac{\nu}{U_m x}} \sim \frac{\nu}{U_m \delta} \quad \delta \frac{d\delta}{dx} \sim \frac{\nu}{U_m}$$

hence it must be

$$\rho g \gamma \frac{\nu}{U_m} \ll \frac{\mu U_m}{\delta^2} \quad \text{or} \quad \frac{g \gamma}{(U_m/\delta)^2} \ll 1$$

which is precisely the assumption made above.

For the present purposes, an approximate representation of the Blasius profile by a straight line and a parabola is used (fig. 1). The profile is to start at the wall ($y = 0$) with a straight piece, which is tangentially joined to a parabolic piece and it, in turn, tangentially onto the constant velocity $U = U_m$.

The result is the approximate representation

$$\left. \begin{aligned} 0 \leq \frac{y}{\delta} \leq 0.175 \quad \frac{U}{U_m} &= 1.68 \frac{y}{\delta} \\ 0.175 \leq \frac{y}{\delta} \leq 1.015 \quad \frac{U}{U_m} &= 1 - \left(1.015 - \frac{y}{\delta}\right)^2 \\ \frac{y}{\delta} = 1.015 \quad \frac{U}{U_m} &= 1 \end{aligned} \right\} \quad (42)$$

earlier indicated for U .

Hence $a = 1.015$ by comparison with equation (23). But for the transitional substitution (equation (29)), U''/U' must be taken according to the exact Blasius series, not $U' = \text{constant}$ and $U'' = 0$ in the linear range and $U'' = \text{constant}$ in the parabolic range. The first terms give

$$\begin{aligned} \frac{U'}{U_m} &= 1.68 \left[1 - 3.65 \left(\frac{y}{\delta} \right)^3 \right] \\ \frac{U''}{U_m} &= -18.4 \left(\frac{y}{\delta} \right)^2 \end{aligned} \quad (43)$$

This supplies all the data necessary for the numerical calculation. Figure 2 shows the previously described polar diagram for a specific c -value ($c = 0.10$), where the several E curves correspond to the written-in values of the stratification quantity K , and L is always equal to zero. For this c -value - say, for $K > \frac{1}{29}$ - there are no intersection points of the E curves with the $\frac{D}{\eta_w}$ curve, hence, no instability. For other c values, this maximum K value, at which instability is still just possible, is different. The highest possible K value at which instability is still just possible (critical value of K) is $K = 1/24$, when assuming $L = 0$. For other values of L ($L = -1/10, +1/10, +1/5$), other critical values of K are obtained.

TABLE 3

TRANSITION POINTS $\alpha\delta^*$ AND $\frac{U_m\delta^*}{v}$ FOR VARIOUS VALUES OFTHE STRATIFICATION QUANTITIES K AND L

$K = -\frac{1}{40}$					$L = 0$	$K = \frac{1}{100}$				
c	$\left(\frac{U_m\delta^*}{v}\right)_1$	$(\alpha\delta^*)_1$	$\left(\frac{U_m\delta^*}{v}\right)_2$	$(\alpha\delta^*)_2$	c	$\left(\frac{U_m\delta^*}{v}\right)_1$	$(\alpha\delta^*)_1$	$\left(\frac{U_m\delta^*}{v}\right)_2$	$(\alpha\delta^*)_2$	
0.20	1.15×10^4	0.038	-----	-----	0.10	1.05×10^5	0.053	4.05×10^5	0.089	
0.30	1.84×10^3	0.078	8.75×10^3	0.187	0.20	7.16×10^3	0.095	2.57×10^4	0.164	
0.35	1.01×10^3	0.101	2.99×10^3	0.232	0.30	1.57×10^3	0.148	3.88×10^3	0.232	
0.375	7.25×10^2	0.121	1.88×10^3	0.251	0.35	9.62×10^2	0.183	1.84×10^3	0.256	
0.40	5.86×10^2	0.139	1.30×10^3	0.266	0.375	7.95×10^2	0.207	1.26×10^3	0.267	
0.45	4.04×10^2	0.193	6.08×10^3	0.282	0.40	7.67×10^2	0.251	8.60×10^2	0.265	
$K = -\frac{1}{60}$					$K = \frac{1}{60}$					
0.20	9.27×10^3	0.050	-----	-----	0.10	1.05×10^5	0.064	2.95×10^5	0.098	
0.30	1.68×10^3	0.093	6.75×10^3	0.202	0.20	7.30×10^3	0.108	2.11×10^4	0.172	
0.35	9.20×10^2	0.121	2.61×10^3	0.240	0.30	1.66×10^3	0.165	3.48×10^3	0.234	
0.375	7.15×10^2	0.140	1.70×10^3	0.257	0.35	1.06×10^3	0.204	1.69×10^3	0.258	
0.40	5.70×10^2	0.159	1.20×10^3	0.271	0.375	8.97×10^2	0.239	1.11×10^3	0.263	
0.45	4.51×10^2	0.233	5.23×10^2	0.264	$K = \frac{1}{40}$					
$K = -\frac{1}{100}$					0.10	1.19×10^5	0.079	2.24×10^5	0.104	
0.20	8.22×10^3	0.060	-----	-----	0.20	8.08×10^3	0.125	1.71×10^4	0.179	
0.30	1.60×10^3	0.106	5.71×10^3	0.211	0.30	1.90×10^3	0.190	3.14×10^3	0.238	
0.35	8.96×10^2	0.136	2.36×10^3	0.245	0.35	1.21×10^3	0.236	1.47×10^3	0.255	
0.375	7.05×10^2	0.156	1.57×10^3	0.262	$K = \frac{1}{30}$					
0.40	5.76×10^2	0.177	1.12×10^3	0.273	0.10	1.42×10^5	0.097	1.68×10^5	0.105	
0.42	5.18×10^2	0.198	8.28×10^3	0.279	0.20	9.04×10^3	0.145	1.44×10^4	0.183	
$K = 0$					0.30	2.10×10^3	0.211	2.73×10^3	0.236	
0.20	7.20×10^3	0.077	3.76×10^4	0.149	0.325	1.69×10^3	0.232	1.90×10^3	0.244	
0.25	3.01×10^3	0.101	1.20×10^4	0.188	$K = \frac{1}{25}$					
0.30	1.53×10^3	0.129	4.64×10^3	0.223	0.20	1.04×10^4	0.166	1.22×10^4	0.177	
0.325	1.15×10^3	0.143	3.29×10^3	0.238	0.225	6.51×10^3	0.181	7.44×10^3	0.195	
0.35	8.93×10^2	0.159	2.07×10^3	0.251	0.23	6.05×10^3	0.186	7.09×10^3	0.199	
0.375	7.36×10^2	0.181	1.42×10^3	0.264	$K = \frac{1}{24}$					
0.40	6.33×10^2	0.205	1.02×10^3	0.274	0.225	7.18×10^3	0.190	-----	-----	
0.42	5.76×10^2	0.239	7.13×10^2	0.273						

TABLE 3 - Continued

$K = -\frac{1}{40}$					$L = -\frac{1}{10}$		$K = 0$		
c	$\left(\frac{U_m \delta^*}{v}\right)_1$	$(\alpha \delta^*)_1$	$\left(\frac{U_m \delta^*}{v}\right)_2$	$(\alpha \delta^*)_2$	c	$\left(\frac{U_m \delta^*}{v}\right)_1$	$(\alpha \delta^*)_1$	$\left(\frac{U_m \delta^*}{v}\right)_2$	$(\alpha \delta^*)_2$
0.30	1.94×10^3	0.072	-----	-----	0.20	8.39×10^3	0.065	-----	-----
0.35	9.94×10^2	0.102	3.16×10^3	0.218	0.30	1.67×10^3	0.113	5.21×10^3	0.207
0.40	6.49×10^2	0.130	1.38×10^3	0.248	0.375	7.94×10^2	0.166	1.51×10^3	0.250
0.425	5.20×10^2	0.155	9.41×10^2	0.254	0.40	6.71×10^2	0.187	1.07×10^3	0.257
0.45	4.58×10^2	0.184	6.38×10^2	0.252					
$K = \frac{1}{60}$									
0.20	1.17×10^4	0.040	-----	-----					
0.25	4.11×10^3	0.062	-----	-----					
0.30	1.81×10^3	0.086	7.46×10^3	0.185					
0.35	1.03×10^3	0.114	2.76×10^3	0.225					
0.40	6.22×10^3	0.150	1.28×10^3	0.253					
0.425	5.29×10^3	0.173	8.63×10^2	0.262					
0.45	4.92×10^3	0.218	5.44×10^2	0.245					

TABLE 3 - Continued

$K = 0$					$L = \frac{1}{10}$	$K = \frac{1}{40}$				
c	$\left(\frac{U_m \delta^*}{v}\right)_1$	$(\alpha \delta^*)_1$	$\left(\frac{U_m \delta^*}{v}\right)_2$	$(\alpha \delta^*)_2$	c	$\left(\frac{U_m \delta^*}{v}\right)_1$	$(\alpha \delta^*)_1$	$\left(\frac{U_m \delta^*}{v}\right)_2$	$(\alpha \delta^*)_2$	
0.20	6.16×10^3	0.096	1.98×10^4	0.187	0.20	8.53×10^3	0.175	1.14×10^4	0.199	
0.30	1.45×10^3	0.139	4.09×10^3	0.242	0.30	1.80×10^3	0.210	2.72×10^3	0.257	
0.375	6.47×10^2	0.195	1.33×10^3	0.286	0.325	1.38×10^3	0.228	1.94×10^3	0.265	
0.40	5.49×10^2	0.214	9.75×10^2	0.296	0.35	1.19×10^3	0.259	1.30×10^3	0.270	
0.425	4.65×10^2	0.251	6.65×10^2	0.300	$K = \frac{1}{35}$					
$K = \frac{1}{60}$					0.20	9.58×10^3	0.188	1.04×10^4	0.195	
0.20	7.12×10^3	0.141	1.49×10^4	0.188	0.25	3.82×10^3	0.203	5.05×10^3	0.225	
0.30	1.57×10^3	0.183	3.10×10^3	0.256	0.30	1.91×10^3	0.224	2.58×10^3	0.252	
0.375	7.86×10^2	0.251	1.07×10^3	0.288	0.325	1.55×10^3	0.246	1.76×10^3	0.262	
0.39	7.66×10^2	0.272	8.45×10^2	0.284	$K = \frac{1}{33}$					
					0.25	4.10×10^3	0.212	4.70×10^3	0.224	
					0.30	2.06×10^3	0.235	2.37×10^3	0.250	

TABLE 3 - Concluded

K = 0					$L = \frac{1}{5}$		$K = \frac{1}{40}$		
c	$\left(\frac{U_m \delta^*}{v}\right)_1$	$(\alpha \delta^*)_1$	$\left(\frac{U_m \delta^*}{v}\right)_2$	$(\alpha \delta^*)_2$	c	$\left(\frac{U_m \delta^*}{v}\right)_1$	$(\alpha \delta^*)_1$	$\left(\frac{U_m \delta^*}{v}\right)_2$	$(\alpha \delta^*)_2$
0.20	5.26×10^3	0.136	1.30×10^4	0.218	0.30	1.77×10^3	0.247	2.26×10^3	0.276
0.30	1.32×10^3	0.157	3.58×10^3	0.269	0.325	1.27×10^3	0.256	1.78×10^3	0.294
0.375	5.52×10^2	0.212	1.21×10^3	0.310	0.35	9.26×10^2	0.268	1.43×10^3	0.308
0.40	4.34×10^2	0.231	9.91×10^2	0.320	0.375	7.52×10^2	0.290	1.04×10^3	0.319
0.45	2.87×10^2	0.272	5.05×10^2	0.340	$K = \frac{1}{37}$				
$K = \frac{1}{60}$					0.30	1.88×10^3	0.258	2.17×10^3	0.280
0.25	3.38×10^3	0.216	4.55×10^3	0.243	0.325	1.35×10^3	0.266	1.71×10^3	0.296
0.30	1.52×10^3	0.212	-----	-----	0.35	1.01×10^3	0.280	-----	-----
0.35	8.65×10^2	0.242	-----	-----	$K = \frac{1}{35}$				
0.375	6.76×10^2	0.259	1.04×10^3	0.317	0.30	2.02×10^3	0.274	-----	-----
0.40	5.39×10^2	0.285	8.17×10^2	0.321	0.325	1.40×10^3	0.274	1.65×10^3	0.294
0.425	5.06×10^2	0.311	5.82×10^2	0.326	0.35	1.04×10^3	0.290	1.26×10^3	0.312

So far, all lengths had been referred to the boundary-layer thickness δ (half the width of the parabola inscribed in the Blasius profile, fig. 1). For the presentation of the mathematical results, the displacement thickness $\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy$ is chosen as reference length ($\delta^* = 0.341\delta$).

The transition points computed for the various stratifications (characterized by K and L) are correlated in table 3, and the corresponding curves plotted in figures 3, 4, 5, and 6. The area circumscribed by the curves represents the area of unstable disturbances. The diagrams indicate the looked-for connection between the wave length of the disturbance $\lambda = 2\pi/\alpha$, the Reynolds number R , and the stratification (K and L) as formulated in equation (4) in the introduction. Each diagram corresponds to a specific value of L ($L = -1/10, 0, +1/10$, and $+1/5$), while the several curves of each diagram relate to different values of K . Each curve is therefore characterized by two parameters K and L . The curve $K = 0, L = 0$ is Tollmien's indifference curve for homogeneous fluid. For stable stratifications ($K > 0, L > 0$), the area of the unstable disturbances is smaller and for unstable disturbances greater than for homogeneous fluids.

The indifference curves themselves exhibit the following unusual fact: For homogeneous fluid and for unstable stratifications each indifference curve extends to infinity at great Reynolds numbers, while for stable stratifications the indifference curves are closed. With increasing stable stratification, the unstable range of the wave length of disturbance enclosed by the indifference curve becomes consistently smaller, until finally at a K value dependent on L the indifference curve shrinks to a point located in the αR plane.

The critical Reynolds number corresponding to a certain state of stratification is of particular interest. In figure 7, the critical Reynolds number $\left(\frac{U_m \delta^*}{\nu}\right)_k$ is plotted against Θ , for several values of the Froude number $\frac{U_m^2}{\delta^* g}$. Thus is the relation important for the comparison with experiment, as formulated in equation (6). For small values of Θ , the variation of the R_k -curves with the Froude number $\frac{U_m^2}{\delta^* g}$ is rather small. But the stability limit Θ_k - that is, the maximum value of the stratification quantity at which turbulence is still possible - still depends to some extent on the Froude number.

TABLE 4
THE CRITICAL REYNOLDS NUMBER $\left(\frac{U_m \delta^*}{\nu}\right)_k$ AS A FUNCTION OF THE RICHARDSON

NUMBER $\Theta = \frac{g}{\rho} \frac{dp}{dy} \left/ \left(\frac{dU}{dy} \right)_w^2 \right.$ AND THE FROUDE NUMBER $\frac{U_m^2}{\delta g}$

$\frac{U_m^2}{\delta g} = 0$		$\frac{U_m^2}{\delta g} = 1$		$\frac{U_m^2}{\delta g} = 2$		$\frac{U_m^2}{\delta g} = 4$		$\frac{U_m^2}{\delta g} = 5$	
Θ	$\left(\frac{U_m \delta^*}{\nu}\right)_k$	Θ	$\left(\frac{U_m \delta^*}{\nu}\right)_k$	Θ	$\left(\frac{U_m \delta^*}{\nu}\right)_k$	Θ	$\left(\frac{U_m \delta^*}{\nu}\right)_k$	Θ	$\left(\frac{U_m \delta^*}{\nu}\right)_k$
-0.0208	387	-0.0200	410	-0.0200	420	-0.0200	440	-0.0200	450
-0.0142	438	-0.0100	480	-0.0100	485	-0.0100	495	-0.0100	500
-0.0087	487	0	575	0	575	0	575	0	575
0	575	0.0100	750	0.0100	740	0.0100	710	0.0100	693
0.0090	738	0.0200	1038	0.0200	1025	0.0200	998	0.0200	980
0.0153	887	0.0250	1287	0.0250	1300	0.0250	1315	0.0250	1305
0.0233	1182	0.0290	1524	0.0290	1620	$\Theta_{crit} = 0.0297$	2350	$\Theta_{crit} = 0.0290$	1850
0.0314	1631	$\Theta_{crit} = 0.0370$	5350	$\Theta_{crit} = 0.0340$	4000				
0.0392	2920								
0.0409	7180								
$\Theta_{crit} = 0.0417$	∞								

The dependence $\Theta_k = \Theta_k \left(\frac{U_m^2}{\delta^* g} \right)$ is reproduced on figure 8. For $\frac{U_m^2}{\delta^* g} = 0$ - hence, for the case that the inertia effect of the stratification is disregarded relative to the gravity effect - it results in $\Theta_k = 0.0409$. Therefore, the inclusion of the friction for nullifying the turbulence affords a considerably lower value of the stratification quantity than without friction, according to Taylor and Goldstein, who obtained $\Theta_k = 0.250$.

By equation (14a) the stratification quantity K is so defined that the velocity gradient must be taken at the critical point which differs according to the position of the critical layer. But, in the present case, it was preferred to form the Richardson number with the velocity gradient at the wall, namely

$$\Theta = -\frac{g}{\rho_0} \frac{d\rho_0}{dy} \bigg/ \left(\frac{dU}{dy} \right)_w^2 \quad (44)$$

as obtained from K by the reduction

$$\Theta = K \left(\frac{dU}{dy} \right)_k^2 \bigg/ \left(\frac{dU}{dy} \right)_w^2$$

Since $\left(\frac{dU}{dy} \right)_k \bigg/ \left(\frac{dU}{dy} \right)_w$ is always close to unity, the Θ values differ little from the K values.

9. COMPARISON OF THEORETICAL WITH EXPERIMENTAL DATA

Velocity and temperature distributions in a flow with stratification of densities, which are very appropriate for the comparison with the present theory, were made some years ago by Reichardt⁴ in the warm-cold air tunnel of the Kaiser Wilhelm-Institute for flow research at Göttingen. The tunnel is 16 m long, 1 m wide, and 25 cm high. The velocity profiles were measured at the end of the tunnel with a hot-wire survey apparatus which, to compensate for the temperature effect, consisted of two hot wires of different diameter. One of these wires could at the same time be used for temperature recording. The study so far included only the stable case, where the upper plate of the tunnel was heated with steam (100°) and the bottom plate cooled by tap water (10° C), hence at a

⁴Not published so far in detail; see the test reports (reference 26).

temperature gradient of 90° C at 25 cm. Very low airspeeds of around 1 m/sec were indicated, since at higher speeds no perceptible effect of the temperature stratification of the flow is produced. Figure 9 represents a velocity and temperature distribution measurement. In the central part of the tunnel where the speed is practically constant, the temperature is also nearly constant.

For the comparison with the present theory, the measured profiles were evaluated to the effect that for each profile the maximum velocity U_m and the boundary-layer thickness δ (and hence the displacement thickness δ^*) of the boundary layer are obtained. With these, the theoretical velocity distribution of the laminar flow was then ascertained and their velocity gradient at the wall

$$\left(\frac{dU}{dy}\right)_w = 1.68 \frac{U_m}{\delta}$$

computed. In figures 10 to 14 are shown for various maximum velocities U_m , the measured velocity and temperature distributions for the boundary layers on the warm and cold plates together with the distributions upon which our stability calculations were based. The relation between temperature and density is, on account of the constant,

$$\frac{\rho}{\rho_w} = \frac{T_w}{T} \quad \text{or} \quad \frac{1}{\rho} \frac{d\rho}{dy} = -\frac{1}{T} \frac{dT}{dy}$$

T , the absolute temperature; subscript w , the value at the wall. The density distribution assumed for the present theory (equation (15)) according to the law

$$\rho_0 = \rho_{0w} e^{-\gamma y}$$

corresponds therefore to a temperature distribution

$$T = T_w e^{\gamma y} \quad (45)$$

The constant γ was determined from the measurements in such a way that the recorded temperature distribution agreed as closely as possible with the distribution given by equation (45), which served as basis of the present calculations (cf. figs. 10, 11, 12, 13, and 14).

With it the Richardson number

$$\Theta = \frac{\gamma g}{\left(\frac{dU}{dy}\right)_w^2} = \frac{\frac{g}{T} \frac{dT}{dy}}{\left(\frac{dU}{dy}\right)_w^2}$$

the Froude number $F^2 = \frac{U_m^2}{\delta g}$ and the Reynolds number $R = \frac{U_m \delta^*}{\nu}$ for each profile can be computed. The value corresponding to the mean temperature in the boundary layer was chosen as the value for the kinematic viscosity ν . The numerical data are reproduced in table 5.

The decision of whether the recorded velocity profile was laminar or turbulent was made from oscillographic records of the voltage fluctuations in the hot wire, but it could also be observed direct from the shape of the velocity profile. Despite the finite tunnel width and the density differences, the measured velocity for the laminar profiles is in good agreement with the Blasius plate flow, while the turbulent profiles have almost the form of the $1/7$ power law.

For comparison, the test points were plotted in an R, Θ diagram, Θ being expressed at logarithmic scale (fig. 15). To each profile in figures 10 to 14, there corresponds one point in this diagram, the laminar and turbulent profile being characterized by different symbols. The solid curve is the theoretical stability limit which gives the critical Reynolds number as function of the stratification quantity Θ for the Froude number $F = 0$. The curves for the other Froude numbers are omitted since they do not differ appreciably from the curve for $F = 0$, according to figure 7. The solid curve gives the boundary between the theoretically stable (laminar) and the theoretically unstable (turbulent) attitudes. The Taylor-Goldstein value $\Theta_{crit} = 0.250$ and the value $R_{crit} = 575$ for plate flow in homogeneous fluid are also included. The comparison of the presented theory with the test data at

which the Froude number $\frac{U_m^2}{\delta g}$ ranges between 0.86 and 3.82 (compare table 5) indicates that in the theoretically stable zone only laminar attitudes, and in the theoretically unstable zone, with one single exception, only turbulent attitudes were observed.⁵ Incidentally, it should be noted that it is not contrary to theory when laminar states are still observed closely above the stability limit $R_{crit} = R_{crit} \Theta$;

⁵From the oscillographic records on the rejected profile, no clear distinction could be made between laminar and turbulent.

TABLE 5.- COMPARISON WITH TEST DATA

Figure No.	$\frac{U_m}{\text{cm sec}^{-1}}$	δ_{cm}	δ^*_{cm}	$\nu_{\text{cm}^2\text{sec}^{-1}}$	$(T_w - T_m)^0$	$\gamma_{\text{cm}^{-1}}$	$\frac{U_m \delta^*}{\nu}$	θ	$F = \frac{U_m^2}{\delta g}$	theoretical	experimental	
14	{w. Pl.	127.5	4.34	1.48	0.202	64	0.0431	934	0.0173	3.82	stable	laminar
	{k. Pl.	134	7.20	2.45	0.151	-25	0.0117	2175	0.0117	2.54	unstable	turbulent
13	{w. Pl.	105	4.87	1.66	0.204	60	0.0359	855	0.0269	2.31	stable	laminar
	{k. Pl.	111	6.13	2.09	0.151	-25	0.0138	1536	0.0144	2.05	unstable	turbulent
12	{w. Pl.	102.5	5.69	1.94	0.204	60	0.0307	975	0.0324	1.88	stable	laminar
	{k. Pl.	104	6.34	2.16	0.155	-32	0.0169	1450	0.0218	1.74	unstable	turbulent
11	{w. Pl.	91	6.38	2.18	0.204	60	0.0274	973	0.0467	1.32	stable	laminar
	{k. Pl.	98.5	6.04	2.06	0.154	-30	0.0165	1317	0.0217	1.64	unstable	partly laminar, partly turbulent
10	{w. Pl.	81	7.78	2.65	0.204	60	0.0225	1052	0.0721	0.86	stable	laminar
	{k. Pl.	87.5	5.75	1.96	0.154	-30	0.0174	1108	0.0261	1.36	stable	laminar

w. Pl. = hot plate k. Pl. = cold plate

for this curve indicates the attitudes where the amplification of the disturbance motion is exactly equal to zero. For an actual appearance of the turbulence, the small disturbances must, however, be considerably amplified; and that is the case only when the Reynolds number is somewhat greater (compare reference 22). The agreement is therefore complete.

The value $\Theta_{crit} = 0.250$ obtained by Taylor and Goldstein is far from the Θ values denoting the boundary between laminar and turbulent flow according to Richardson's measurements. It is true that Taylor's and Goldstein's stability investigations had been based upon different velocity profiles than realized here in experiment. The presented theory, on the other hand, whose assumptions are rather closely satisfied by the measurements, indicates a very satisfactory agreement with the experiment, according to figure 15.

Translated by J. Vanier
National Advisory Committee
for Aeronautics

REFERENCES

1. Richardson, L. F.: Turbulence and Vertical Temperature Difference near Trees. Phil. Mag. 49, p. 81, 1925.
2. Taylor, G. I.: Turbulence. Journ. of the Meteorological Society, vol. 53, p. 201, 1927.
3. Bjerknes, V., Bjerknes, J., Solberg, H., and Bergeron, T.: Physikalische Hydrodynamik. Berlin 1933.
4. Taylor, G. I.: An Experiment on the Stability of Superposed Streams of Fluid. Proc. Cambr. Phil. Soc., vol. 23, p. 730, 1927.
5. Schmidt, W.: Der Massenaustausch in freier Luft und verwandte Erscheinungen (Probleme der kosmischen Physik). Hamburg 1925.
6. Prandtl, L.: Einfluss stabilisierender Kräfte auf die Turbulenz. Vorträge aus dem Gebiete der Aerodynamik und verwandter Gebiete. Aachen 1929. p. 1.
7. Prandtl, L.: "Über die ausgebildete Turbulenz. Vhdlg. des II. Intern. Kongr. f. techn. Mechanik, Zürich 1926.
8. Prandtl, L.: Meteorologische Anwendung der Strömungslehre. "Beiträge zur Physik der freien Atmosphäre" (Bjerknes-Festschrift), vol. 19, p. 188, 1932.
9. Richardson, L. F.: The Supply of Energy from and to Atmospheric Eddies. Proc. Roy. Soc. London, vol. 97, p. 354 (1920).
10. Taylor, G. I.: Internal Waves and Turbulence in a Fluid of Variable Density. Rapp. Proc.-Verb. Cons. Intern. p. 1'Expl. de la Mer, LXVII. Kopenhagen, pp. 20 to 30 (1931).
11. Taylor, G. I.: Effect of Variation in Density on the Stability of Superposed Streams of Fluid. Proc. Roy. Soc. A. Vol. 132, p. 499, 1931.
12. Goldstein, S.: On the Stability of Superposed Streams of Fluids of Different Densities. Proc. Roy. Soc. A. Vol. 132, p. 524, 1931.
13. Tollmien, W.: Über die Entstehung der Turbulenz. Nachr. d. Ges. d. Wiss. zu Göttingen, Math.-Phys. Klasse 1929, p. 21, and Verhandlg. d. III. Intern. Kongr. f. techn. Mech. Stockholm 1930, p. 105.

14. Rayleigh, Lord: On the Stability or Instability of Certain Fluid Motions III. Proc. London Math. Soc. 27, 1895.
15. Sommerfeld, A.: Ein Beitrag zur hydrodynamischen Erklärung der turbulenten Flüssigkeitsbewegung. Atti d. IV. Congr. int. dei Mathem. Rome 1909.
16. Mises, R. v.: Beitrag zum Oszillationstheorem, Heinrich Weber-Festschrift 1912; derselbe: Zur Turbulenztheorie, Jahresber. d. deutsch. Mathem. Ver. 1912.
17. Hopf, L.: Der Verlauf kleiner Schwingungen in einer Strömung reibender Flüssigkeit. Ann. d. Phys. 44, p. 1, 1914, and: Zur Theorie der Turbulenz. Ann. d. Phys. 59, p. 538, 1919.
18. Prandtl, L.: Bemerkungen über die Entstehung der Turbulenz. Ztschr. f. angew. Math. u. Mech. vol. 1, p. 431, 1921.
19. Tietjens, O.: Beiträge zum Turbulenzproblem. Diss. Göttingen 1922 and. Ztschr. f. angew. Math. u. Mech. vol. 5, p. 200, 1915.
20. Wendt, F.: Turbulente Strömungen zwischen zwei rotierenden coaxialen Zylindern. Ing.-Arch. vol. IV, p. 577, 1933.
21. Schlichting, H.: Über die Entstehung der Turbulenz in einem rotierenden Zylinder. Nachr. d. Ges. d. Wiss. zu Göttingen, Math.-Phys. Klasse 1932, p. 160.
22. Schlichting, H.: Zur Entstehung der Turbulenz bei der Plattenströmung. Nachr. d. Ges. d. Wiss. zu Göttingen, Math.-Phys. Klasse, 1933, p. 181. also: Ztschr. f. angew. Math. u. Mech. vol. 13, 1933.
23. Blasius, H.: Grenzschichten in Flüssigkeiten mit kleiner Reibung. Diss. Göttingen 1907 and Ztschr. f. Math. u. Phys. vol. 51, p. 1, 1908.
24. Prandtl, L.: Über Flüssigkeitsbewegung bei sehr kleiner Reibung. Verhdlg. d. III. Int. Math.-Kongr. Heidelberg 1904, or "Vier Abhandlungen zur Hydrodynamik und Aerodynamik, Göttingen 1927.
25. Prandtl, L.: Modellversuche und theoretische Studien über die Turbulenz einer geschichteten Luftströmung. Deutsche Forschung No. 14, Berlin 1930.
26. Prandtl, L., and Reichardt, H.: Einfluss von Wärmeschichtung auf die Eigenschaften einer turbulenten Strömung. Deutsche Forschung No. 21, Berlin 1934.

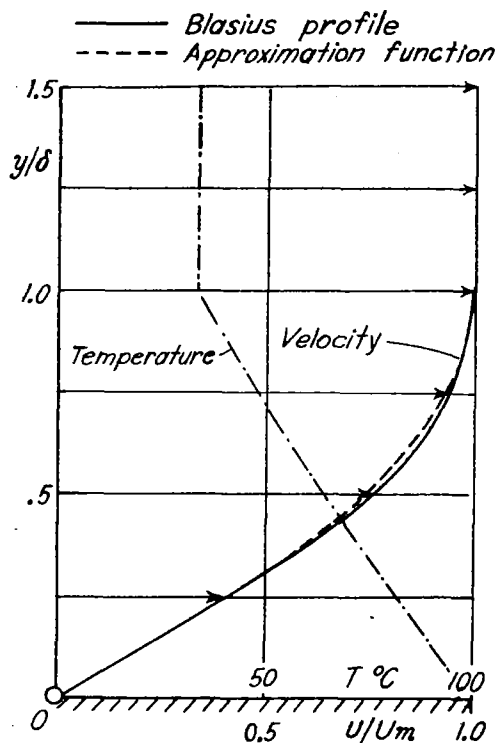


Figure 1.- The investigated laminar flow with stratification of densities.

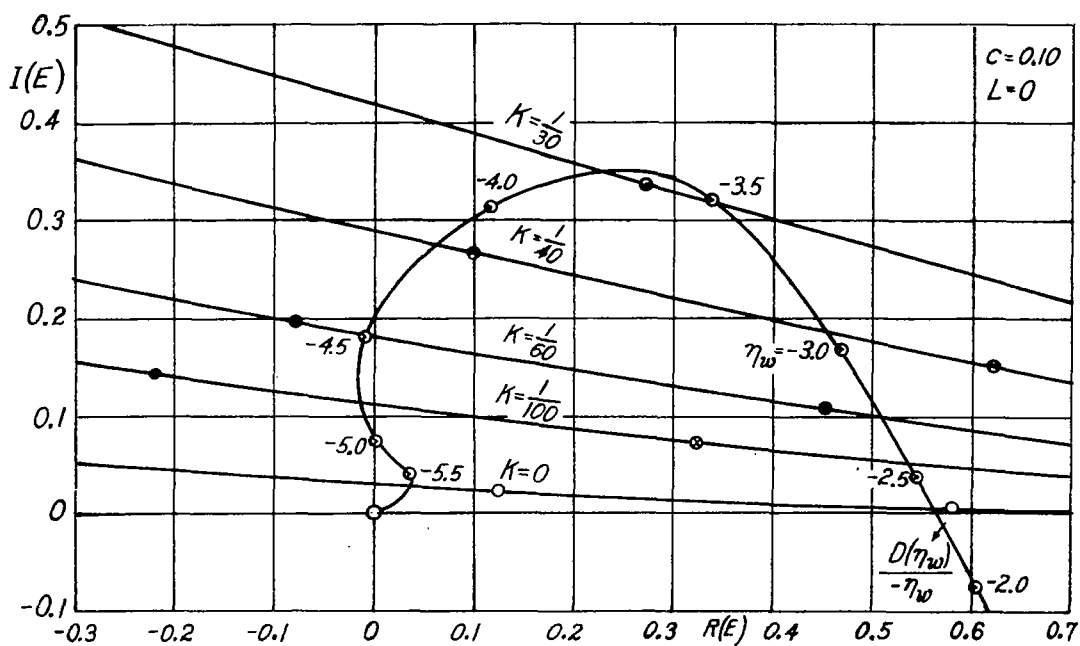


Figure 2.- Polar diagram $\frac{D}{\eta_w}$ curves and $E(\alpha, c, K, L)$ curves for $c = 0.10$ and $K = 0, \frac{1}{100}, \frac{1}{60}, \frac{1}{40}, \frac{1}{30}$; $L = 0$.

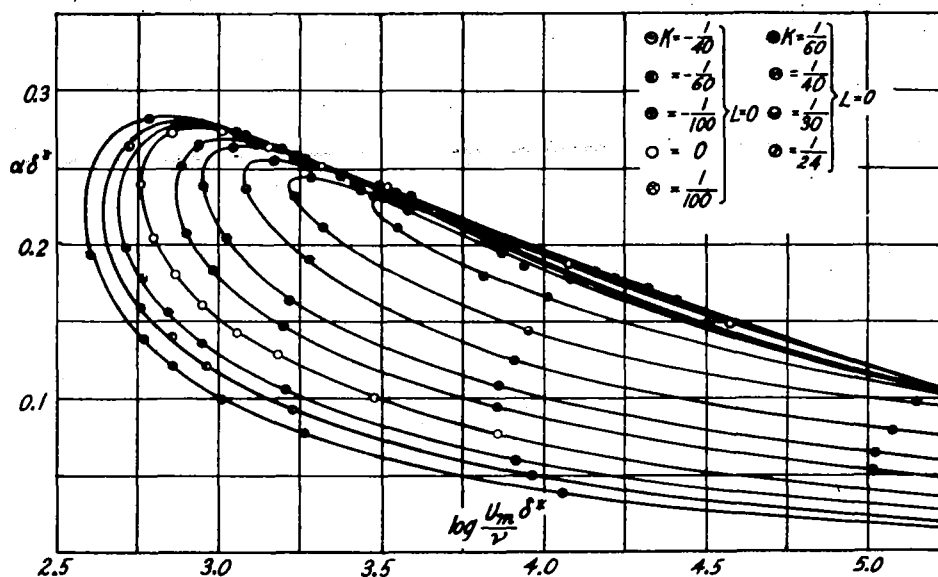


Figure 3.- The indifference curves for the plate flow with stratification of densities. The reciprocal of the disturbance wave length $\alpha\delta^*$ as a function of the Reynolds number $\frac{U_m \delta^*}{\nu}$ and the stratification measure K , for $L = 0$.

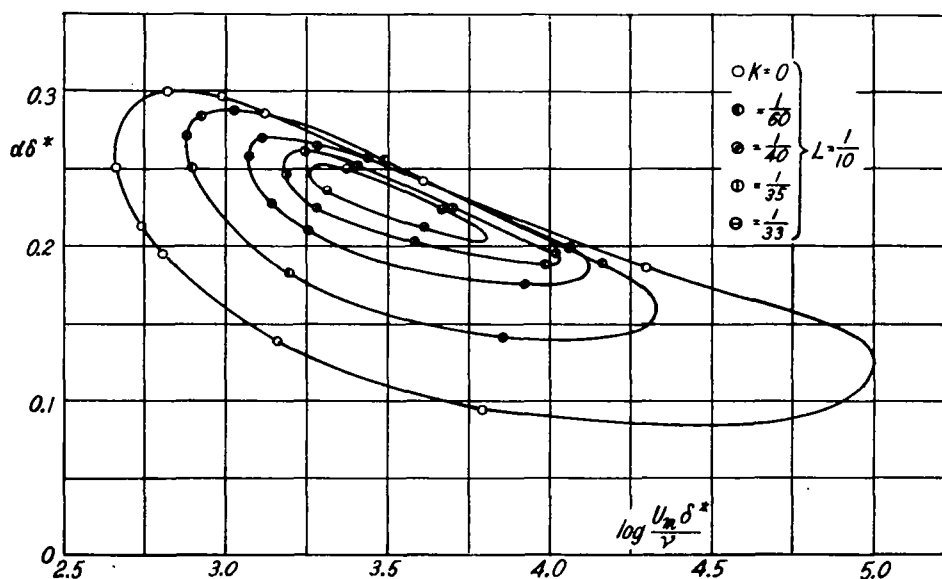


Figure 4.- The indifference curve for the plate flow with stratification of densities. The reciprocal of the disturbance wave length $\alpha\delta^*$ as a function of the Reynolds number $\frac{U_m \delta^*}{\nu}$ and the stratification measure K , for $L = \frac{1}{10}$.

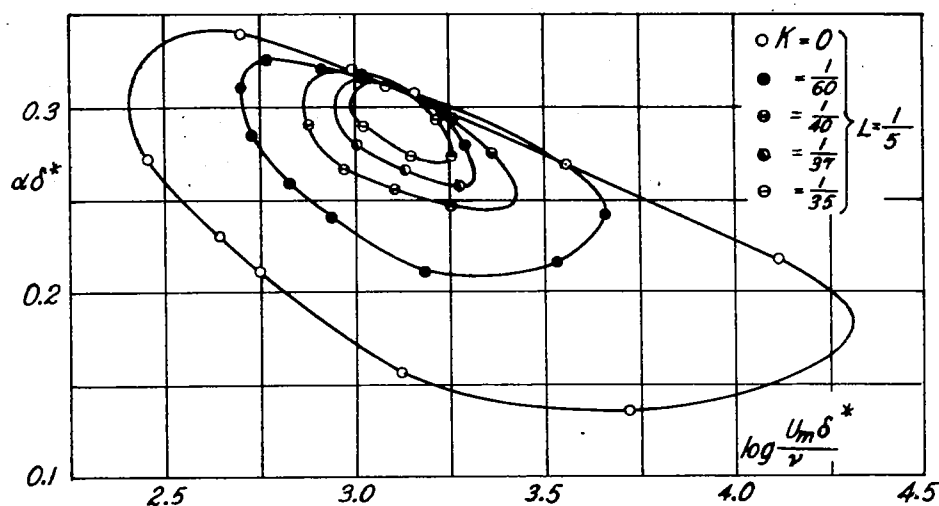


Figure 5.- The indifference curve for the plate flow with stratification of densities. The reciprocal of the disturbance wave length $\alpha\delta^*$ as a function of the Reynolds number $\frac{U_m \delta^*}{\nu}$ and the stratification measure K , for $L = \frac{1}{5}$.

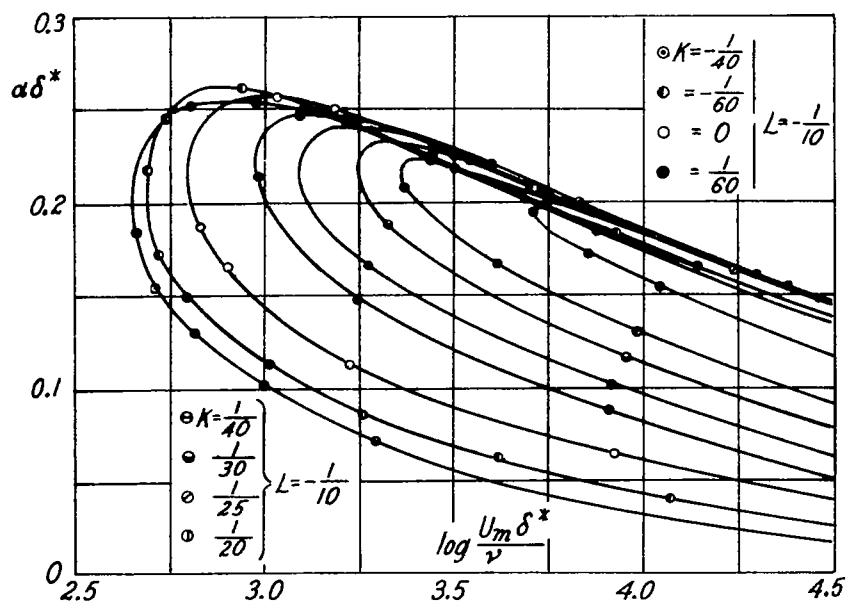


Figure 6.- The indifference curve for the plate flow with stratification of densities. The reciprocal of the disturbance wave length $\alpha\delta^*$ as a function of the Reynolds number $\frac{U_m \delta^*}{\nu}$ and the stratification measure K , for $L = -\frac{1}{10}$.

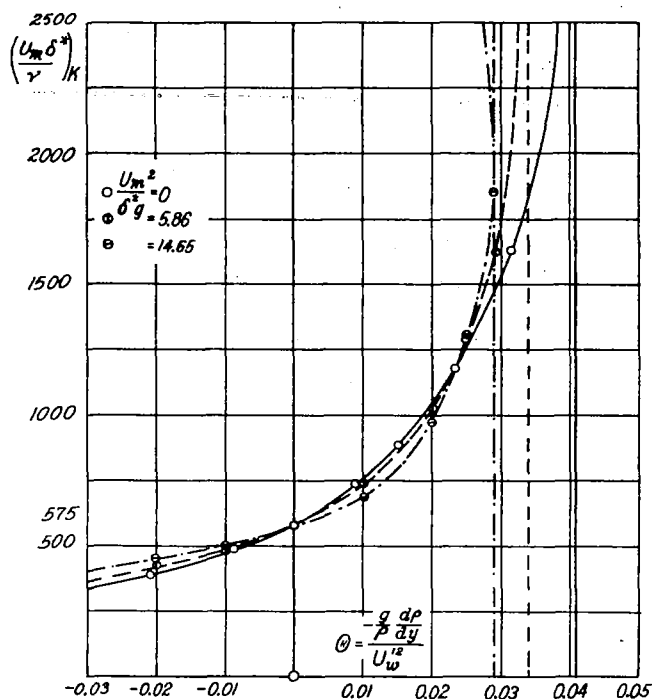


Figure 7.- The critical Reynolds number of the plate flow as a function of the Richardson number and the Froude number $R_k = R_k \left(\Theta, \frac{U_m^2}{\delta^* g} \right)$.

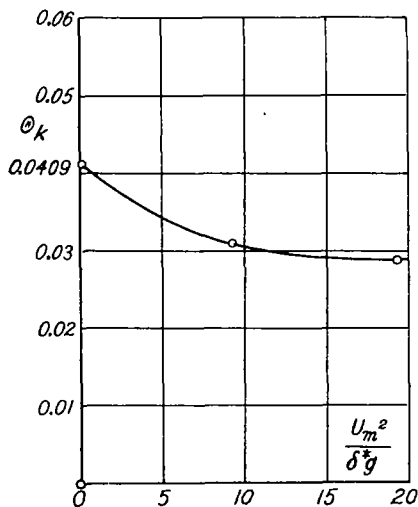


Figure 8.- Stability limit Θ_k as a function of Froude number $\frac{U_m^2}{\delta^* g}$.

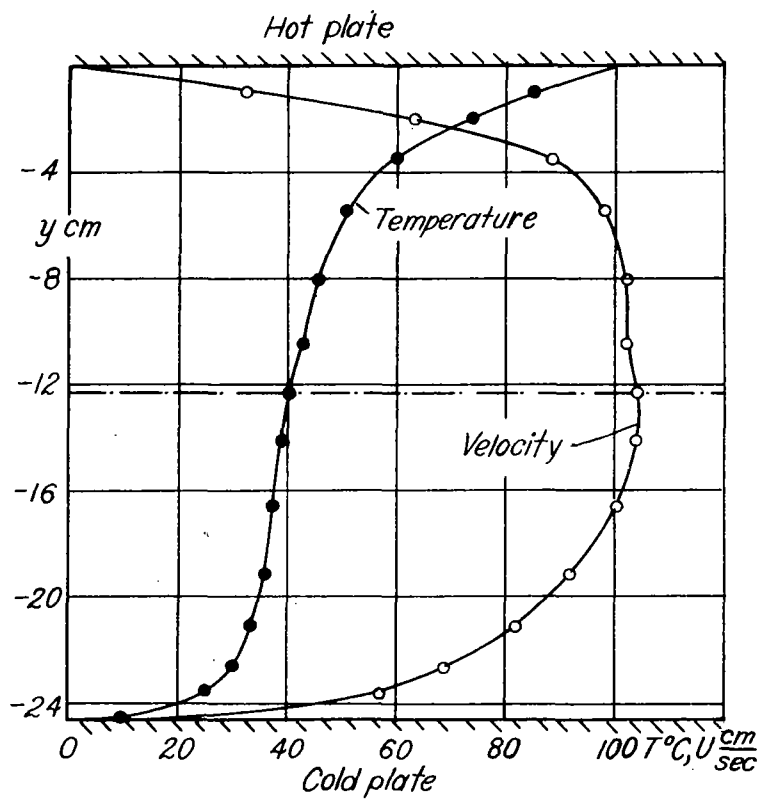


Figure 9.- A velocity and temperature distribution in the Göttingen hot-cold air tunnel (according to Reichardt).

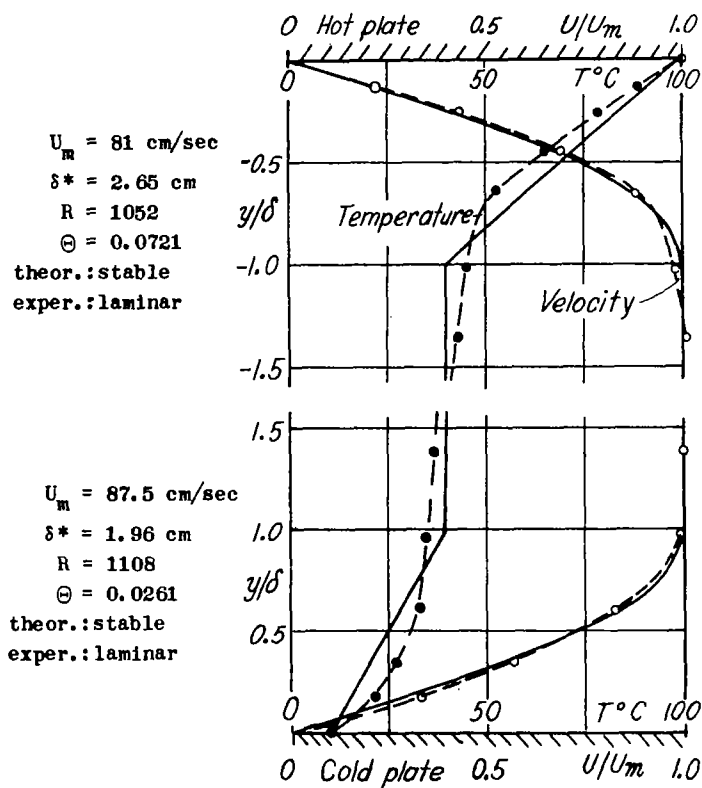


Figure 10.

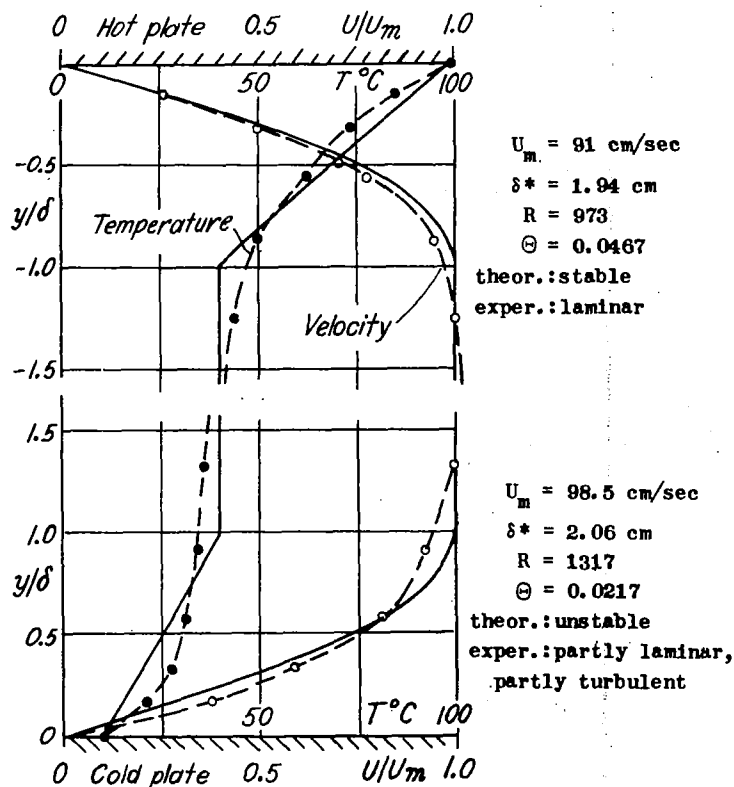


Figure 11.

Figures 10 through 14.- Comparison with test data (velocity profiles). The dashed curves were measured; the solid curves served as basis for the stability calculations. Agreement between measurement and calculation exists when "theor.:stable" is combined with "exper.:laminar," or "theor.:unstable" with "exper.:turbulent."

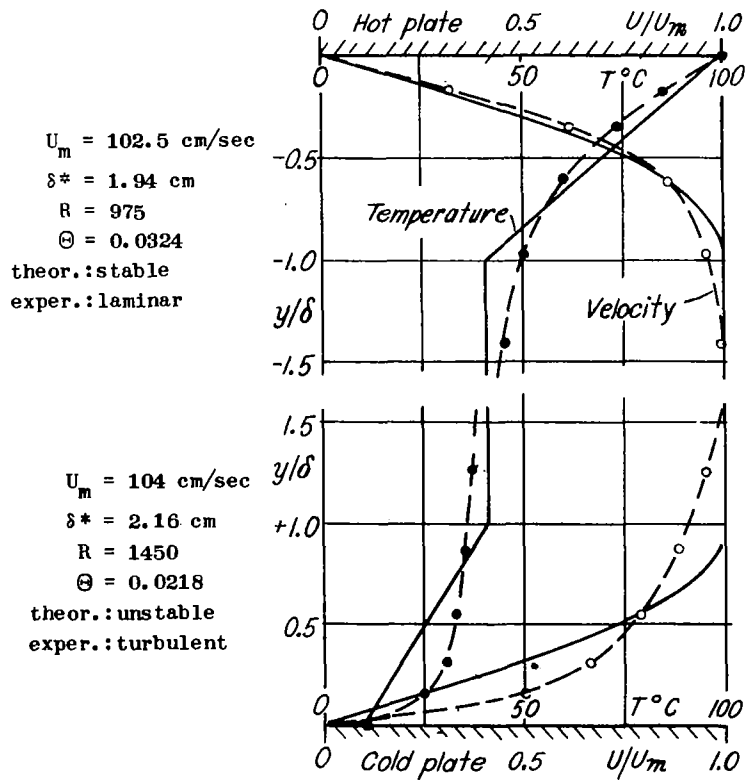


Figure 12.

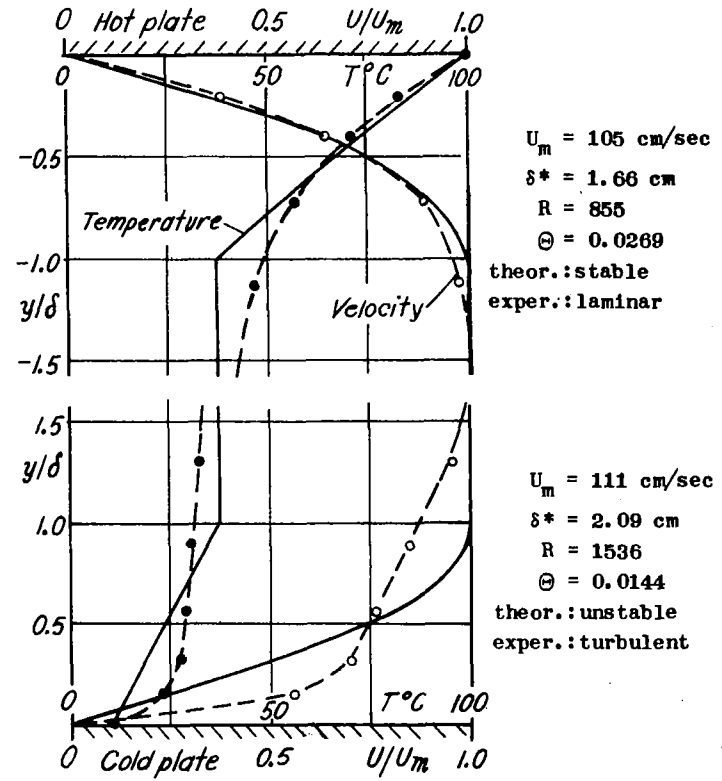


Figure 13.

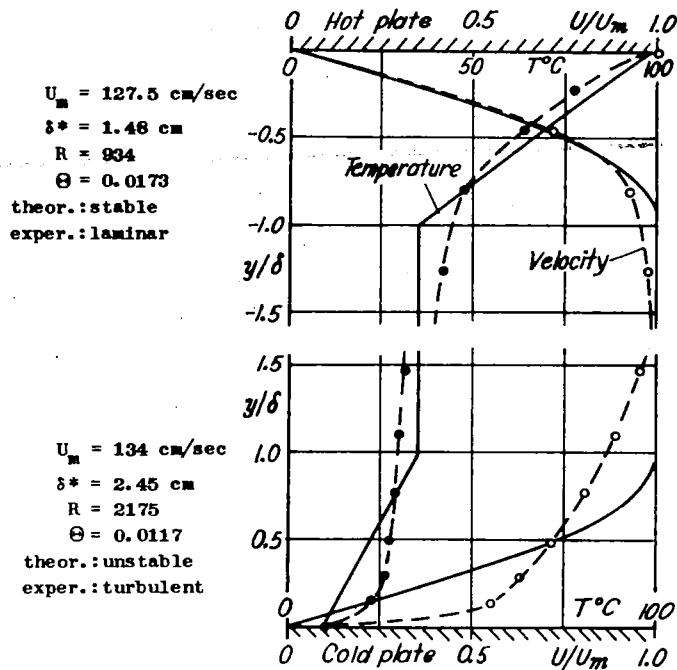


Figure 14.

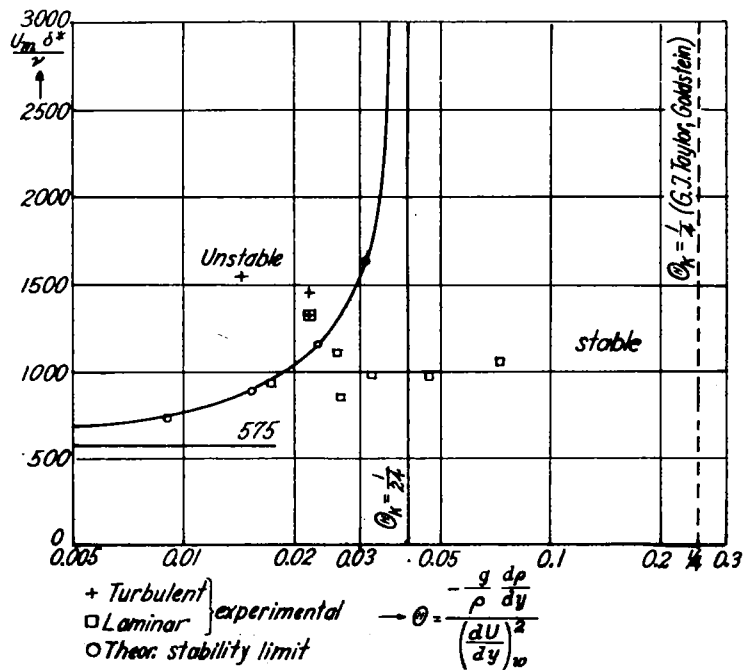


Figure 15.- Comparison with test data. In agreement with theory, only laminar states were observed in the stable range, and only turbulent states in the unstable one.

NASA Technical Library



3 1176 01441 1764